Derivation and Closure of Baer and Nunziato Type Multiphase Models by Averaging a Simple Stochastic Model

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DERIVATION AND CLOSURE OF BAER AND NUNZIATO TYPE MULTIPHASE MODELS BY AVERAGING A SIMPLE STOCHASTIC MODEL

VINCENT PERRIER* AND ENRIQUE GUTIÉRREZ †

Abstract. In this article, we show how to derive a multiphase model of Baer and Nunziato type with a simple stochastic model. Baer and Nunziato models are known to be unclosed, namely, they depend on modeling parameters, as interfacial velocity and pressure, and relaxation terms, whose exact expression is still an open question. We prove that with a simple stochastic model, interfacial and relaxation terms are equivalent to the evaluation of an integral, which cannot be explicitly computed in general. However, in different particular case matching with a large range of applications (topology of the bubbles/droplets, or special flow regime conditions), the interfacial and relaxation parameters can be explicitly computed, leading to different models that are either nonlinear versions or slight modifications of previously proposed models. The validity domains of previously proposed models are clarified, and some modeling parameters of the averaged system are linked with the local topology of the flow. Last, we prove that usual properties like entropy dissipation are ensured with the new closures found.

1. Introduction.

The Baer and Nunziato model was first presented in [5] in the context of granular compressible multiphase flows. In [42], the system was reformulated as follows:

$$\begin{aligned}
&\partial_t \alpha_k + \mathbf{u}_I \cdot \nabla \alpha_k = \mu(P_k - P_{\bar{k}}) \\
&\partial_t\big(\alpha_k \rho_k\big) + \mathrm{div}_{\mathbf{x}}\big(\alpha_k \rho_k \mathbf{u}_k\big) = 0, \\
&\partial_t\big(\alpha_k \rho_k \mathbf{u}_k\big) + \mathrm{div}_{\mathbf{x}}\big(\alpha_k \left(\rho_k \mathbf{u}_k \otimes \mathbf{u}_k + P_k\right)\big) = P_I \nabla \alpha_k + \lambda(\mathbf{u}_{\bar{k}} - \mathbf{u}_k), \\
&\partial_t\big(\alpha_k \rho_k E_k\big) + \mathrm{div}_{\mathbf{x}}\Big(\alpha_k \left(\rho_k E_k + P_k\right) \mathbf{u}_k\Big) = P_I \mathbf{u}_I \cdot \nabla \alpha_k - \mu P_I (P_k - P_{\bar{k}}) + \lambda \mathbf{u}_I \cdot (\mathbf{u}_{\bar{k}} - \mathbf{u}_k),
\end{aligned} \tag{1.1}$$

where k is the index of the phase ($k = 1$ or $k = 2$). In (1.1), $\bar{k}$ denotes the conjugate index of k, i.e. $\bar{k} = 2$ if $k = 1$, and $\bar{k} = 1$ if $k = 2$. α_k denotes the volume fraction of the phase k, ρ_k its density, P_k its pressure, $\mathbf{u}_k$ its velocity, and E_k its total energy, which is defined as

$$E_k = \varepsilon_k + \frac{|\mathbf{u}_k|^2}{2},$$

where ε_k is the internal specific energy. It is linked with the other thermodynamic parameters by an equation of state

$$\varepsilon_k = \varepsilon_k(P_k, \rho_k).$$

System (1.1) depends on interfacial terms: the interfacial velocity $\mathbf{u}_I$ and the interfacial pressure P_I, which must be defined. Last, system (1.1) depends on relaxation parameters λ and μ, which shall be defined too. System (1.1) with $\lambda = \mu = 0$ will be called "the hyperbolic part" (even if we are not sure yet that the system is hyperbolic), and the hyperbolic part with, moreover, $P_I = 0$ and $\mathbf{u}_I = 0$ will be called "the conservative part."

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A large body of work exists on proposing closures for $\mathbf{u}_I$ and P_I, suitable for different applications. In the original work of Baer and Nunziato [5], the closure $\mathbf{u}_I = \mathbf{u}_1$ and $P_I = P_2$ was proposed when one of the media is nearly incompressible and was obtained by physical modeling arguments. It has been widely used in the literature, for instance [30, 20, 43, 13]. The other classical option consists in imposing mathematical properties on the system, for example, obtaining a linearly degenerate field for $\mathbf{u}_I$, as was proposed by Coquel et al. [10]. This closure sets the interfacial velocity to $\mathbf{u}_I = a\mathbf{u}_1 + (1-a)\mathbf{u}_2$, where $a = \alpha_1\rho_1/\rho$ and $\rho = \alpha_1\rho_1 + \alpha_2\rho_2$. In a similar way, the interfacial pressure is set to $P_I = bP_1 + (1-b)P_2$, where $b = (1-a)T_2/T$ and $T = aT_1 + (1-a)T_2$, T_k being the temperature of the phase k. See [18, 23, 34] for examples of its use. Finally, the so-called *acoustic approximation* is also commonly chosen to close the system (1.1). See, for example, [8] for a description of the formulation and applicability of this closure.

Some generalizations of the closure presented in [10] have been proposed. Coquel et al. [11] introduced a slight modification in the calculation of a and ρ, i.e. $a = \chi\alpha_1\rho_1/\rho$ and $\rho = \chi\alpha_1\rho_1 + (1-\chi)\,\alpha_2\rho_2$, where χ is a nondimensional scalar lying in $[0,1]$ that characterizes the flow regime [33]. In like manner, Guillemaud [23] introduced up to four free parameters in the expressions of a and b, associated with thermodynamic coefficients, in order to take into account differences in the turbulent structure.

A large number of works present closures based on averaging procedures, showing a successful behavior for some particular applications. They are based on setting $\mathbf{u}_I = \omega_1^u\mathbf{u}_1 + \omega_2^u\mathbf{u}_2$ and $P_I = \omega_1^P P_1 + \omega_2^P P_2$, where ω_k^u and ω_k^P are weights that vary depending on the formulation. The most used closure of this group is the one proposed by Saurel et al. [42], where $\omega_k^u = \alpha_k\rho_k/\rho$ and $\omega_k^P = \alpha_k$. Other authors proposed different values of the aforementioned weights, such as Delahaye et al. [15] ($\omega_k^u = \omega_k^P = \alpha_k$), Chen et al. [7] ($\omega_k^u = \omega_k^P = \alpha_{\bar{k}}$, especially designed in a Rayleigh-Taylor instability context), Lhuillier [31] ($\omega_k^u = 1/2$ and $\omega_k^P = \alpha_k$, in a context of phase transition), and Ransom et al. [38] ($\omega_k^u = \omega_k^P = 1/2$). Linga et et al. [32] and Jin et al. [29] proposed more complex weights, but followed the same general idea. Some advances have been made in reference to the closure of the system (1.1) when dealing with more than two phases, i.e. $k = 1, \ldots, N$. The multiphase extension of the Baer and Nunziato closure can be written as $\mathbf{u}_I = \mathbf{u}_1$ and $P_I = \sum_{k=2}^{N} P_k$. This is the most used closure in multicomponent scenarios (see, for instance, [24, 25]). More complicated closures, still involving a single interfacial velocity, but involving several interfacial pressures have been proposed also (see [26]). Belonging to this more sophisticated modeling, it is worth highlighting Muller et al. recent work [36], where different closures are studied in multiphase flows, ensuring meaningful physical properties such as a nonnegative entropy production and thermodynamic stability. In addition to these, other closures for multicomponent scenarios can be found [9, 41].

To our knowledge, the first attempt of using the Drew and Passman averaging procedure for obtaining a full closed system was in [2], even if this procedure was used for directly defining a numerical scheme, without caring about the continuous system solved. By computing the limit when the mesh size converges to 0, a first attempt at finding appropriate closure was proposed in [8]. However, the closure proposed is not invariant by a Galilean frame change. This closure was corrected in [17] by using a stochastic modeling approach. Nevertheless, in [17], very restrictive hypothesis were made on the stochastic process, which induced to not finding any relaxation term, and to finding a simple closure for the pressure and velocity interfaces. In this paper,

we propose to compute a formal system including

- a more complex model of pressure and velocity interface depending on the local averaged topology of the flow,
- relaxation terms depending also on the local topology of the flow.

This article is organized as follows. section 2 is dedicated to recalling the derivation of Drew and Passman. This derivation depends on an *averaging* procedure which is not defined in general. In section 3, we define a simple stochastic model based on a Gaussian process for the flow topology which will be used as a basis for the averaging procedure. If the choice of a Gaussian process allows us to perform explicit computations, its physical relevance is more questionable, and this is why a significant part of this section is dedicated to the link between the properties of the Gaussian process and the local topology of the flow. Then in section 4, properties on the stochastic model are derived for allowing an averaging of the terms found in section 2. This leads to a formal derivation of the relaxation and nonconservative terms of (1.1) depending on an integral. Then in section 5, we explore different topologies and flow regimes which allow us to explicitly compute the integral found. The most general system is an extension of (1.1) and reads

$$\begin{aligned}
&\partial_t \alpha_k + \mathbf{u}_I \cdot \nabla \alpha_k = \mathcal{R}_k^{(\alpha)},\\
&\partial_t\big(\alpha_k\rho_k\big) + \mathrm{div}_{\mathbf{x}}\big(\alpha_k\rho_k\mathbf{u}_k\big) = 0,\\
&\partial_t\big(\alpha_k\rho_k\mathbf{u}_k\big) + \mathrm{div}_{\mathbf{x}}\big(\alpha_k\left(\rho_k\mathbf{u}_k\otimes\mathbf{u}_k + P_k\right)\big) = \mathbf{P}_I\nabla\alpha_k + \mathcal{R}_k^{(\rho\mathbf{u})},\\
&\partial_t\big(\alpha_k\rho_k E_k\big) + \mathrm{div}_{\mathbf{x}}\Big(\alpha_k\left(\rho_k E_k + P_k\right)\mathbf{u}_k\Big) = (P\mathbf{u})_I\cdot\nabla\alpha_k + \mathcal{R}_k^{(\rho E)},
\end{aligned} \tag{1.2}$$

where $\mathbf{P}_I$ is a tensor and $(P\mathbf{u})_I$ is not directly linked with P_I and $\mathbf{u}_I$. In section 6, classical properties such as hyperbolicity and entropy dissipation are proved. The last section is a conclusion.

2. Derivation of Baer and Nunziato type model by the method of [16].

The aim of this section is to show how Baer and Nunziato models can be derived with the Drew and Passman method [16].

We point out here that we are interested in multi*phase* flows, and not in multi*component* flows: in multi*component* flows, the different fluids are supposed to share the same volume, each one of them being characterized by its density and its partial pressure (e.g., mixture of different gases, or of miscible liquids). In multi*phase* flows, each phase has its own volume.

2.1. Averaged system.

For the moment, we suppose that a sufficiently regular *level set* function f exists such that the fluid k is in $\mathbf{x}$ if and only if $f(\mathbf{x}) \geq 0$. The indicator function of the fluid k is therefore defined as

$$\chi_k(\mathbf{x}) = \mathbf{1}_{\{f(\mathbf{x})\geq 0\}}.$$

The indicator function χ_k ensures the following equation weakly:

$$\partial_t\chi_k + \mathbf{v}_i\cdot\nabla\chi_k = 0. \tag{2.1}$$

Indeed, (2.1) is clearly ensured when $\chi_k = 0$ or $\chi_k = 1$ because both the gradient and time derivative vanish and holds also on a jump of χ_k in a weak sense provided $\mathbf{v}_i$ is the local interface velocity. For each phase, the Euler system of equation, abbreviated as

$$\partial_t\mathbf{U}_k + \mathrm{div}_{\mathbf{x}}\mathbf{F}_k(\mathbf{U}_k) = 0, \tag{2.2}$$

holds inside each phase. Following [16], the following equation holds for each phase in a weak sense:

$$\chi_k \left(\partial_t \mathbf{U}_k + \mathrm{div}_{\mathbf{x}} \mathbf{F}_k(\mathbf{U}_k)\right) = 0. \tag{2.3}$$

Indeed, (2.3) holds when the fluid is not present, because $\chi_k = 0$, but also when the fluid is present, because then the Euler system holds.

Based on (2.1) and (2.3), the following system can be formally derived:

$$\partial_t \left(\chi_k \mathbf{U}_k\right) + \mathrm{div}_{\mathbf{x}}(\chi_k \mathbf{F}_k(\mathbf{U}_k)) = \left(\mathbf{F}_k(\mathbf{U}_k) - \mathbf{v}_i \mathbf{U}_k\right) \nabla \chi_k. \tag{2.4}$$

Note that by defining $\hat{\mathbf{F}} = (0, \mathbf{F})$ and $\hat{\mathbf{U}} = (1, \mathbf{U})$, (2.1) and (2.4) can be rewritten as

$$\partial_t \left(\chi_k \hat{\mathbf{U}}_k\right) + \mathrm{div}_{\mathbf{x}} \left(\chi_k \hat{\mathbf{F}}_k(\hat{\mathbf{U}}_k)\right) = \left(\hat{\mathbf{F}}_k(\hat{\mathbf{U}}_k) - \mathbf{v}_i \hat{\mathbf{U}}_k\right) \nabla \chi_k. \tag{2.5}$$

One idea developed in [2, 17] is that provided the local topology is known and sufficiently regular, $\mathbf{v}_i$ can be computed by solving a one dimensional Riemann problem. Indeed, at one interface (so, one point in which $f(\mathbf{x}) = 0$), and if we suppose that $\nabla f(\mathbf{x})$ does not vanish, the unit normal to the interface can be defined as

$$\mathbf{n} = \frac{\nabla f(\mathbf{x})}{\|\nabla f(\mathbf{x})\|},$$

and $\mathbf{n}$ is inward with respect to the set $\left\{\mathbf{x} \in \mathbb{R}^d \quad f(\mathbf{x}) \geq 0\right\}$. The Riemann problem in the direction $\mathbf{n}$ with phase $\bar{k}$ on the left and phase k on the right can be solved, and denoting by $u^\star_{\bar{k}k}(\mathbf{n})$ and $P^\star_{\bar{k}k}(\mathbf{n})$ the velocity and pressure of the contact discontinuity, the interface velocity is $\mathbf{v}_i = \mathbf{u}^\star_{\bar{k}k}(\mathbf{n}) = u^\star_{\bar{k}k}(\mathbf{n})\mathbf{n}$, and the right hand side of (2.5) can be computed as follows:

$$\begin{aligned}\left(\hat{\mathbf{F}}_k(\hat{\mathbf{U}}_k) - \mathbf{v}_i \hat{\mathbf{U}}_k\right) \nabla \chi_k &= \hat{\mathbf{F}}_k(\hat{\mathbf{U}}^\star_k) \nabla f(\mathbf{x}) - \hat{\mathbf{U}}^\star_k \mathbf{v}_i \cdot \nabla f(\mathbf{x}) \\ &= \left(\hat{\mathbf{F}}_k(\hat{\mathbf{U}}^\star_k)\mathbf{n} - \hat{\mathbf{U}}^\star_k u^\star_{\bar{k}k}(\mathbf{n})\right) \|\nabla f(\mathbf{x})\|.\end{aligned}$$

Denoting by $\rho^\star_k$ and $E^\star_k$ the density and total energy at the interface, we find

$$\begin{aligned}\hat{\mathbf{F}}_k(\hat{\mathbf{U}}^\star_k)\mathbf{n} - \hat{\mathbf{U}}^\star_k u^\star_{\bar{k}k} &= \left(\begin{pmatrix} 0 \\ \rho^\star_k \mathbf{u}^\star_{\bar{k}k} \cdot \mathbf{n} \\ \rho^\star_k \mathbf{u}^\star_{\bar{k}k} u^\star_{\bar{k}k}(\mathbf{n}) + P^\star_{\bar{k}k}(\mathbf{n})\mathbf{n} \\ (\rho^\star_k E^\star_k + P^\star_{\bar{k}k}(\mathbf{n})) u^\star_{\bar{k}k}(\mathbf{n}) \end{pmatrix} - u^\star_{\bar{k}k}(\mathbf{n}) \begin{pmatrix} 1 \\ \rho^\star_k \\ \rho^\star_k \mathbf{u}^\star_{\bar{k}k} \\ \rho^\star_k E^\star_k \end{pmatrix}\right) \\ &= \begin{pmatrix} -u^\star_{\bar{k}k}(\mathbf{n}) \\ 0 \\ P^\star_{\bar{k}k}(\mathbf{n})\mathbf{n} \\ P^\star_{\bar{k}k}(\mathbf{n}) u^\star_{\bar{k}k}(\mathbf{n}) \end{pmatrix}.\end{aligned}$$

Defining the Lagrangian flux by

$$\hat{\mathbf{F}}^{lag}_{\bar{k}k}(\mathbf{n}) := \begin{pmatrix} -u^\star_{\bar{k}k}(\mathbf{n}) \\ 0 \\ P^\star_{\bar{k}k}(\mathbf{n})\mathbf{n} \\ P^\star_{\bar{k}k}(\mathbf{n}) u^\star_{\bar{k}k}(\mathbf{n}) \end{pmatrix}$$

gives the final expression for (2.5)

$$\partial_t \left(\chi_k \hat{\mathbf{U}}_k\right) + \mathrm{div}_{\mathbf{x}} \left(\chi_k \hat{\mathbf{F}}_k(\hat{\mathbf{U}}_k)\right) = \hat{\mathbf{F}}^{lag}_{\bar{k}k} \left(\frac{\nabla f(\mathbf{x})}{\|\nabla f(\mathbf{x})\|}\right) \|\nabla f(\mathbf{x})\|. \tag{2.6}$$

2.2. Averaging. For obtaining the set of equations at the macroscopic scale, it remains to average (2.6). We consider an averaging operator $\mathbb{E}[\cdot]$. This operator is supposed to commute with the spatial and time derivation (referred as Gauss and Leibniz rules in [16, p. 102]). Then

$$\partial_t\left(\mathbb{E}\left[\chi_k\hat{\mathbf{U}}_k\right]\right)+\mathrm{div}_{\mathbf{x}}\left(\mathbb{E}\left[\chi_k\hat{\mathbf{F}}_k(\hat{\mathbf{U}}_k)\right]\right)=\mathbb{E}\left[\hat{\mathbf{F}}^{lag}_{\bar{k}k}\left(\frac{\nabla f(\mathbf{x})}{\|\nabla f(\mathbf{x})\|}\right)\|\nabla f(\mathbf{x})\|\right]. \tag{2.7}$$

Provided $\hat{\mathbf{U}}$ is supposed to not depend on the flow topology, the left hand side of (2.7) is equal to the time derivative and the conservative part of (1.1)

$$\partial_t\left(\alpha_k\hat{\mathbf{U}}\right)+\mathrm{div}_{\mathbf{x}}\left(\alpha_k\hat{\mathbf{F}}_k(\hat{\mathbf{U}}_k)\right)=\mathbb{E}\left[\hat{\mathbf{F}}^{lag}_{\bar{k}k}\left(\frac{\nabla f(\mathbf{x})}{\|\nabla f(\mathbf{x})\|}\right)\|\nabla f(\mathbf{x})\|\right], \tag{2.8}$$

whereas the right hand side

$$\mathbb{E}\left[\hat{\mathbf{F}}^{lag}_{\bar{k}k}\left(\frac{\nabla f(\mathbf{x})}{\|\nabla f(\mathbf{x})\|}\right)\|\nabla f(\mathbf{x})\|\right] \tag{2.9}$$

includes the nonconservative and relaxation terms of (2.7), which cannot be further made explicit. Usually, at this point, (2.9) is modeled by convective and relaxation terms depending on parameters, which can be guessed by physical arguments or by trying to impose mathematical properties. Our point of view in this article is strongly different: we want to make (2.9) further explicit, and this requires the definition of the averaging operator. For defining it, we propose to use an explicit stochastic model for χ_k, which will be developed in the next section.

3. Stochastic modeling of two phase flow. Now, following [16], we aim at *averaging* the system obtained in (2.4) with an *ensemble averaging*.

We aim at defining a distribution of the two phases of the flow, based on random modeling. We suppose that the presence of the fluid k is defined by a stochastic process. In order to be able to perform explicit computations, we suppose that the distribution is the image of a Gaussian stochastic process $g_{\mathbf{x}}$:

$$\chi_k(\mathbf{x})=\frac{1+\mathrm{sgn}(g_{\mathbf{x}})}{2}.$$

Note that we then have

$$\chi_{\bar{k}}(\mathbf{x})=\frac{1-\mathrm{sgn}(g_{\mathbf{x}})}{2}=\frac{1+\mathrm{sgn}(-g_{\mathbf{x}})}{2},$$

so that if χ_k is the Heaviside image of the Gaussian process, $\chi_{\bar{k}}$ is the Heaviside image of the opposite Gaussian process.

3.1. Consistency of the model with the known moment of the flow. In (1.1), the presence of each fluid is characterized by the volume fraction, which must

be equal to $\alpha_k(\mathbf{x})$. Denoting by $m_k(\mathbf{x})$ the mean of this stochastic process, we have

$$\begin{aligned}
\alpha_k(\mathbf{x}) &= \mathbb{E}\left[\frac{1+\operatorname{sgn}(g_{\mathbf{x}})}{2}\right] \\
&= \frac{1}{\sqrt{2\pi}}\int_0^{\infty} \exp(-(u-m_k(\mathbf{x}))^2/2)\,\mathrm{d}u \\
&= \frac{1}{\sqrt{2\pi}}\int_{-m_k(\mathbf{x})}^{\infty} \exp(-u^2/2)\,\mathrm{d}u \\
&= \frac{1}{\sqrt{\pi}}\int_{-m_k(\mathbf{x})/\sqrt{2}}^{\infty} \exp(-v^2)\,\mathrm{d}v \\
\alpha_k(\mathbf{x}) &= \frac{1}{2}\left(1+\operatorname{erf}\left(\frac{m_k(\mathbf{x})}{\sqrt{2}}\right)\right).
\end{aligned}$$

Thus, for a given $\alpha_k(\mathbf{x})$, $m_k(\mathbf{x})$ is given by

$$m_k(\mathbf{x}) = \sqrt{2}\,\operatorname{erf}^{-1}(2\alpha_k(\mathbf{x})-1).$$

Moreover, computing the derivative of this last relation gives

$$\nabla\alpha_k(\mathbf{x}) = \frac{\exp(-m_k(\mathbf{x})^2/2)}{\sqrt{2\pi}}\,\nabla m_k(\mathbf{x}).$$

Because of the relation $\alpha_{\bar{k}}(\mathbf{x}) = 1-\alpha_k(\mathbf{x})$, we immediately find $m_{\bar{k}}(\mathbf{x}) = -m_k(\mathbf{x})$ and $\nabla m_{\bar{k}}(\mathbf{x}) = -\nabla m_k(\mathbf{x})$.

3.2. Auto-correlation function and its physical interpretation. We denote by R the auto-correlation function defined by

$$R\,:\,(\mathbf{x},\mathbf{y})\in\mathbb{R}^d\times\mathbb{R}^d \longmapsto \mathbb{E}\left[(g_{\mathbf{x}}-m_k(\mathbf{x}))(g_{\mathbf{y}}-m_k(\mathbf{y}))\right].$$

Moreover, the stochastic process is supposed to be such that $\operatorname{Var}(g_{\mathbf{x}}) = 1$ for all $\mathbf{x}$. As g is a Gaussian process, the vector $(g_{\mathbf{x}}, g_{\mathbf{y}})$ is a Gaussian vector, which is totally determined by its mean $(m_k(\mathbf{x}), m_k(\mathbf{y}))$ and its covariance matrix

$$\begin{pmatrix} 1 & R(\mathbf{x},\mathbf{y}) \\ R(\mathbf{x},\mathbf{y}) & 1 \end{pmatrix}.$$

We remark that

$$\begin{aligned}
R(\mathbf{x},\mathbf{y}) &= \mathbb{E}\left[(g_{\mathbf{x}}-m_k(\mathbf{x}))(g_{\mathbf{y}}-m_k(\mathbf{y}))\right] \\
&= \mathbb{E}\left[g_{\mathbf{x}}g_{\mathbf{y}}\right] - \left(\mathbb{E}\left[g_{\mathbf{x}}\right]\right)^2 \\
&= \mathbb{E}\left[(-g_{\mathbf{x}})\,(-g_{\mathbf{y}})\right] - \left(\mathbb{E}\left[(-g_{\mathbf{x}})\right]\right)^2,
\end{aligned}$$

which means that the auto-correlation function is the same for $g_{\mathbf{x}}$ and $(-g_{\mathbf{x}})$, so that an index k or $\bar{k}$ is useless.

In the next sections, we will see that

$$\partial_{\mathbf{xy}} R(\mathbf{x},\mathbf{x}) \tag{3.1}$$

will be an important parameter in the homogenized system. In this section, We aim at giving a physical sense to (3.1).

Given a normalized direction $\mathbf{n}$, we can consider $(g_{\mathbf{x}+t\mathbf{n}})_{t\in\mathbb{R}}$, which is also a Gaussian process, for which the auto-correlation function is

$$R_{\mathbf{n}}(t_x,t_y) = R(\mathbf{x}+t_x\mathbf{n}, \mathbf{x}+t_y\mathbf{n}),$$

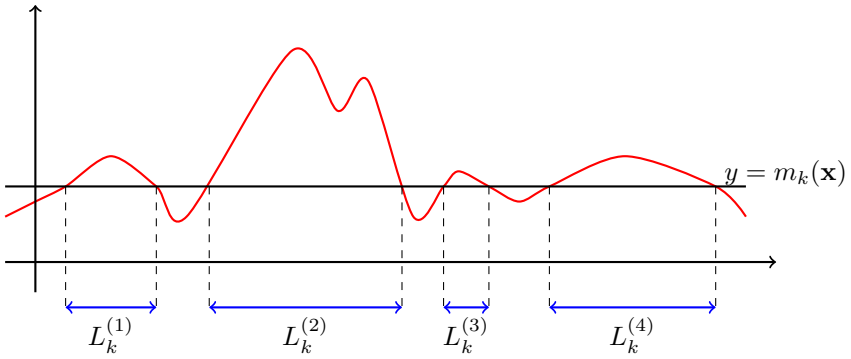

FIG. 3.1. *Example of one realization of the Gaussian process (in red). The values of x such that the Gaussian process is above $m_k(\mathbf{x})$ belong to the phase k, and the values of x for which $g_{\mathbf{x}} \leq m_k(\mathbf{x})$ belong to the phase $\bar{k}$. The double arrows in blue link an upcrossing of the value $m_k(\mathbf{x})$ with the first next downcrossing of $m_k(\mathbf{x})$. The length $L_k^{(i)}$ can be understood as the length of bubbles (or droplets) of the phase k.*

so that

$$\partial_{t_x t_y} R_{\mathbf{n}}(0,0) = \mathbf{n}^T \partial_{\mathbf{xy}} R(\mathbf{x},\mathbf{x})\mathbf{n}. \tag{3.2}$$

All the numerical experiments will be performed in one dimension, and the extension to higher dimension will be done using (3.2). In the one dimensional case, an example of simulation is drawn in Figure 3.1.

3.2.1. Bubble size distribution. By studying sets of realizations of the Gaussian process $g_{\mathbf{x}}$, some insights into the nature of the process can be extracted. First, by using the model presented above, the distribution f_{L_1} of the sizes of the inclusions L_1 (i.e. bubbles) can be obtained. It can be obtained by computing several realizations of the Gaussian process $g_{\mathbf{x}}$. For each realization, when the phase 1 is present at the midpoint of the domain, the distance from this midpoint to the left phase change point L_1 is measured. Figure 3.2 shows the typical shape of the bubble distribution (i.e. its normalized density spectrum) when using the proposed model. In this case, a *squared exponential* auto-correlation function has been used [39]:

$$R_{\mathrm{SE}}(\mathbf{x},\mathbf{y}) = \exp\left(-\frac{(\mathbf{y}-\mathbf{x})^2}{2\zeta^2}\right), \tag{3.3}$$

where ζ is a free parameter, set to 3.5 in this case. Results shown in Figure 3.2 highlight the typical behavior of f_{L_1} when using the model presented above: a single maximum value is obtained for a particular bubble length, with a constant decrease to the right and left of that value. Note, therefore, that the model is not suitable for flows involving more than one typical bubble size.

3.2.2. Example of oceanic flows. When it comes to studying real applications, if the shape of the bubble distribution f_{L_1} is known, in general it is possible to design an auto-correlation function that gives rise to the desired spectrum. There are many different auto-correlation functions available in the literature [39], which in general depend on free parameters to control the influence range and the shape of the function. In this way, real distributions can be reproduced by properly choosing the auto-correlation function and its free parameters.

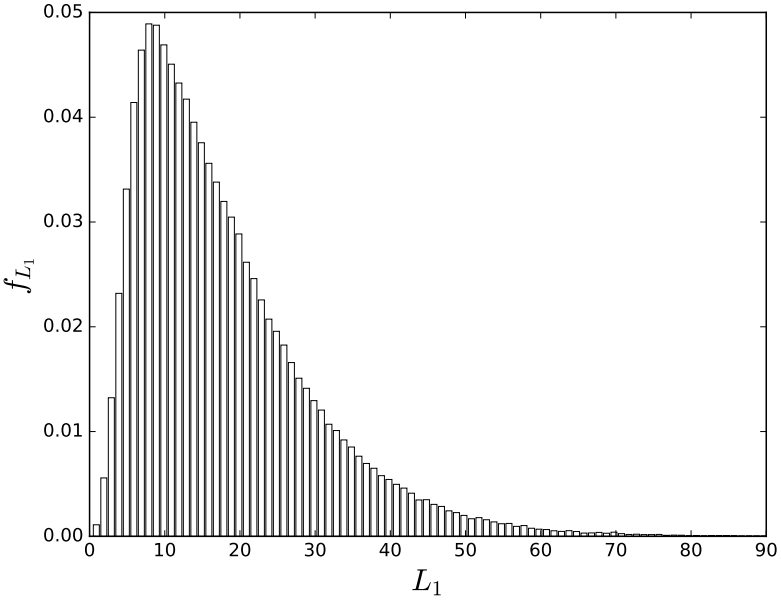

FIG. 3.2. *Typical bubble size distribution obtained by using the model described in the present paper.* f_{L_1} *is the normalized probability density function, and* L_1 *is the bubble characteristic length. A* squared exponential *auto-correlation function has been used here, with a parameter* ζ *equal to* 3.5 *(see* (3.3)*).*

As an example of this process, we analyze here the case of the bubble distribution in the first meters of the ocean, which has been widely studied due to its intrinsic importance in many engineering applications [19]. In this case, the bubble spectra was completely characterized by Deane et al. [14], identifying two different regions. These regions are separated at a characteristic length scale called the *Hinze scale* [27], whose value for this particular application is one millimeter. The bubble spectrum shows a well-defined $-3/2$ power-law scaling with the bubble characteristic length L_1 to the left of the Hinze scale, and a $-10/3$ power-law scaling to the right [14]. As shown in Figure 3.3, the model presented here is capable of reproducing the theoretical behavior and the experimental results. In this case, a γ-*exponential* auto-correlation function [39] has been employed, $R_\gamma(\mathbf{x},\mathbf{y}) = \exp\left(-\left((\mathbf{y}-\mathbf{x})/\zeta\right)^{\upsilon}\right)$, where ζ and υ are free parameters, set for this particular case to 0.05 and 0.5, respectively.

3.2.3. Average bubble length. The average size of the bubbles is a particularly important property of the distribution. Henceforth we focus on the *Matérn* auto-correlation function $R_{\mathcal{M}}$, scaled as [39]:

$$R_{\mathcal{M}}(\mathbf{x},\mathbf{y}) = \frac{2^{1-\upsilon}}{\Gamma(\upsilon)}\left(\frac{\sqrt{2\upsilon}\,(\mathbf{y}-\mathbf{x})}{\zeta}\right)^{\upsilon}\mathcal{K}_\upsilon\left(\frac{\sqrt{2\upsilon}\,(\mathbf{y}-\mathbf{x})}{\zeta}\right) \tag{3.4}$$

with positive parameters ζ and υ, where $\mathcal{K}_\upsilon$ is a modified Bessel function [3]. In particular, we focus in the case $\upsilon = 3/2$. We define λ as the square root of the second order derivative of the auto-correlation function. For the particular case considered

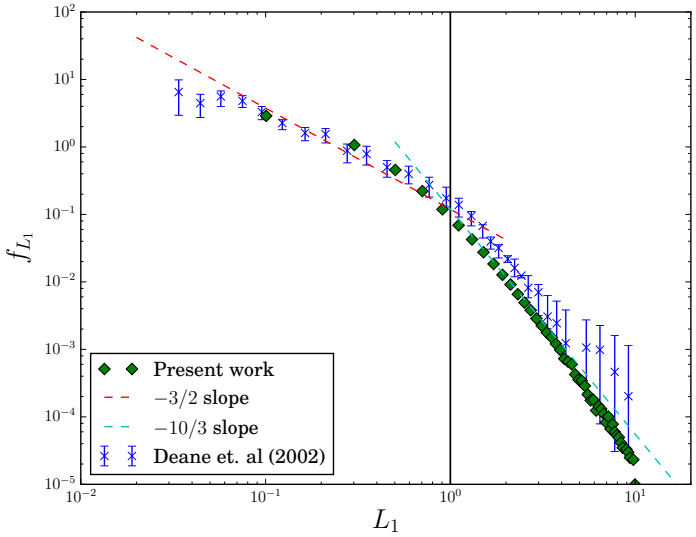

FIG. 3.3. *Typical bubble spectrum in oceanic flows applications, in log scale. It has been obtained by using a* γ-exponential *auto-correlation function* R_γ. *Results obtained with our model are compared against those of Deane et al. [14], for a volume fraction* α *equal to* 0.1. *The horizontal axis shows the bubble characteristic length (in mm), and the vertical one is the bubble distribution* f_{L_1}. *Dashed lines show the theoretical values for the slopes to the right and to the left of the Hinze scale. The vertical solid line indicates the position of the reference Hinze scale (*1*mm). For this particular application, the parameters* υ *and* ζ *of the* γ-exponential *auto-correlation function are set to* 0.05 *and* 0.5, *respectively.*

here (i.e. $R_{\mathcal{M},\upsilon=3/2}$), λ can be expressed as follows:

$$\lambda_{\mathcal{M},\upsilon=3/2} = \frac{\sqrt{3}}{\zeta}. \tag{3.5}$$

Following [4], the number of upcrossings N_u of the value u of a one dimensional stationary Gaussian stochastic process in $[0,1]$ follows the Rice formula [40]

$$\mathbb{E}\left[N_u\right] = \frac{\lambda}{2\pi}\,\exp\left(-\frac{u^2}{2}\right),$$

where λ is the square root of the second spectral moment, matching with the square root of the second order derivative of the auto-correlation function. In our case, we are interested in the upcrossings with the value $m_k(\mathbf{x})$. Referring to Figure 3.1, the total length of fluid k is

$$\sum_{i=1}^{N_{m_k(\mathbf{x})}} L_k^{(i)},$$

and we expect the mean of this to be $\alpha_k(\mathbf{x})$. If we suppose that the number of upcrossings and the length are independent, then

$$\mathbb{E}\left[L_k\right]\mathbb{E}\left[N_{m_k(\mathbf{x})}\right] = \mathbb{E}\left[\sum_{i=1}^{N_{m_k(\mathbf{x})}} L_k^{(i)}\right] = \alpha_k(\mathbf{x}),$$

which gives the following heuristic formula for $\mathbb{E}[L_k]$:

$$\mathbb{E}[L_k] \approx \hat{L}_k := \frac{\alpha_k(\mathbf{x})}{\mathbb{E}\left[N_{m(\mathbf{x})}\right]} = \frac{2\pi}{\exp\left(-\frac{m_k(\mathbf{x})^2}{2}\right)}\,\alpha_k(\mathbf{x})\,\frac{1}{\lambda}. \tag{3.6}$$

Numerical simulations have been conducted to evaluate the deviations of $\hat{L}_k$ defined in (3.6) with respect to the experimental values $\mathbb{E}[L_k]$. By using (3.4), several realizations of the Gaussian stochastic process have been computed, resulting in different sets of $\mathbb{E}[L_1]$ values, as explained in subsubsection 3.2.1. For each value of α, the number of realizations is set to $5 \cdot 10^6/\alpha$, where the α dividing factor is intended to ensure an approximately constant number of measures for the different volume fractions. Additionally, the domain size is computed as $C_d \cdot \hat{L}_1$. Here $\hat{L}_1$ is the expected bubble length given by (3.6), and C_d is a constant that ensures that less than 1% of the bubbles cross the domain boundaries (typically $C_d \approx 50$). Finally, the cell size is obtained by dividing the expected bubble length $\hat{L}_1$ into a constant number of points, i.e. $\hat{L}_1/C_r$, where C_r is set to 40 in this case.

Figure 3.4 shows a comparison between the average bubble lengths obtained numerically and those obtained by using (3.6), with respect to the volume fraction α. Several ζ parameters have been considered, ranging from 2.6 to 4.6 in 0.4 increments. Correspondingly, Figure 3.5 presents the relative error among results, showing the suitability of (3.6) around $\alpha = 0.4$ for the chosen conditions.

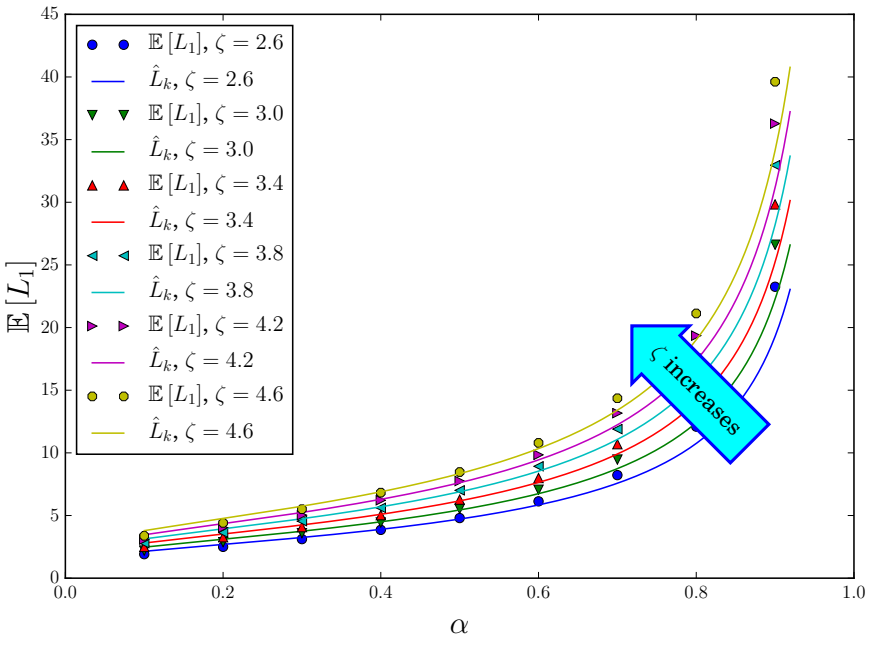

FIG. 3.4. *Results comparison of the average characteristic size of the bubble* $\mathbb{E}[L_1]$ *with respect to the volume fraction* α*. Numerical results are represented by points and those obtained by using* (3.6) *by lines. Simulations have been conducted using a* Matérn *auto-correlation function,* $R_{\mathcal{M},\upsilon=3/2}$*. Several* ζ *parameters have been considered, ranging from* 2.6 *to* 4.6 *in* 0.4 *increments.*

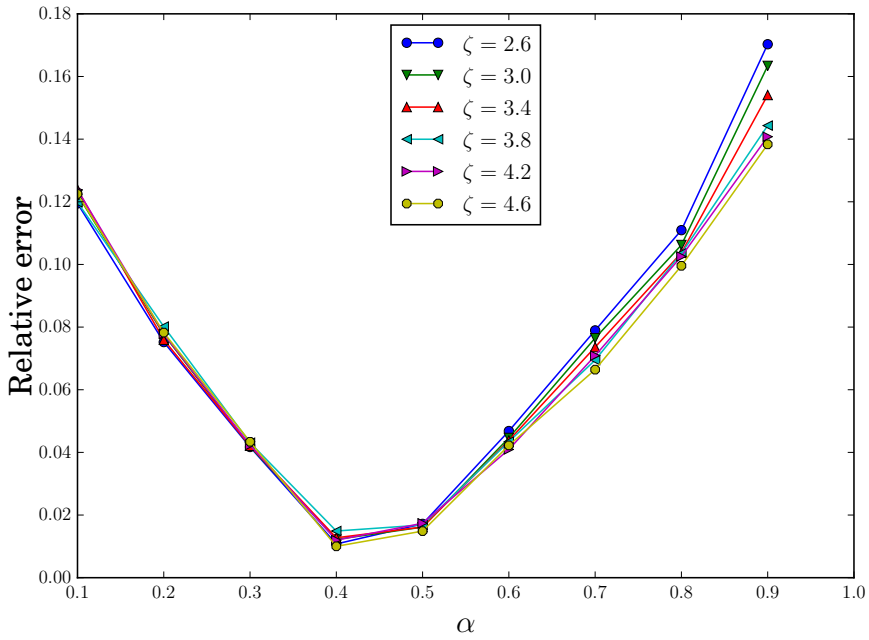

FIG. 3.5. *Error plot of the results comparison presented in Figure 3.5. The relative error* E_1 *has been computed as* $E_1 = \left|\mathbb{E}[L_1] - \hat{L}_1\right| / \mathbb{E}[L_1]$*, where* $\mathbb{E}[L_1]$ *denotes the average bubble length computed numerically, and* $\hat{L}_1$ *theoretically.*

3.2.4. Dependency of $\mathbb{E}[L_1]$ on λ. In any case, it seems that the average bubble size remains proportional to λ^{-1} for any value of α, as (3.6) suggests. For example, using the same auto-correlation function as above for a volume fraction ranging from 0.1 to 0.9, the mean bubble sizes $\mathbb{E}[L_1]$ obtained for different λ values have been computed. Note that a change in λ is obtained by changing ζ and keeping v constant, as extracted from (3.5). Figure 3.6 shows the behavior of the computed $\mathbb{E}[L_1]$ with respect to λ, in log scale, showing a well defined slope of -1.

This behavior of $\mathbb{E}[L_1]$ with respect to λ is very interesting in multiple dimensions. Indeed, following (3.2),

$$\lambda_{R,\mathbf{n}} = \sqrt{\mathbf{n}^T \partial_{\mathbf{xy}} R(\mathbf{x},\mathbf{x})\mathbf{n}}.$$

As a consequence, in multiple dimensions, it is not possible to get the average radius of the bubbles, but $\partial_{\mathbf{xy}} R(\mathbf{x},\mathbf{x})$ fully defines the anisotropy of the average bubbles: its eigendirections define the directions of anisotropy, whereas the ratio of the square roots of its eigenvalues give the anisotropy intensity.

4. Formal derivation of the nonconservative and relaxation terms. The aim of this section is to define additional properties on $g_{\mathbf{x}}$ for giving a sense to (2.9) when f is no more a regular function, but a stochastic process. Equation (2.9) includes the derivation of an indicator function, and this indicator is the image of f by the Heaviside function H. Given any regular function f, the gradient of $H(f(\mathbf{x}))$ in the distribution sense is

$$\delta_{\{f(x)=0\}} \nabla f. \tag{4.1}$$

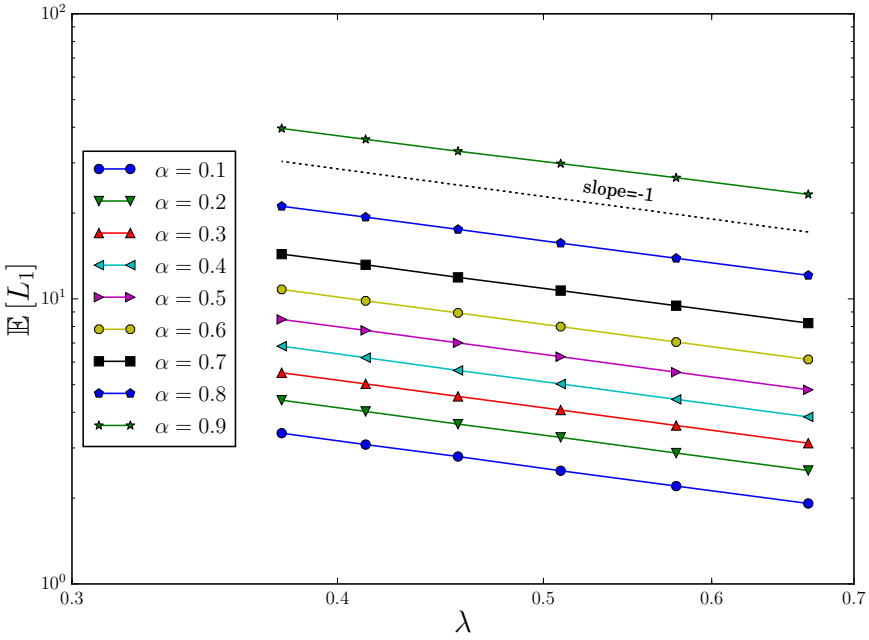

FIG. 3.6. *Average bubble length $\mathbb{E}[L_1]$ varying with respect to the square of the second order derivative of the auto-correlation function λ, for several values of the volume fraction α. A well-defined slope of -1 can be observed for all values of α.*

Now, we are interested in the expression (4.1) in the case when f is no more deterministic, but stochastic. What we see from (4.1) is that we must be interested in the joint behavior of $g_{\mathbf{x}}$ and its gradient. In subsection 4.1, we will derive the properties of $g_{\mathbf{x}}$ and $\nabla g_{\mathbf{x}}$. Then in subsection 4.2, the constraints on $g_{\mathbf{x}}$ will be derived for ensuring the measurability of (4.1) when f is equal to $g_{\mathbf{x}}$.

4.1. Moments of $g_{\mathbf{x}}$ and $\nabla g_{\mathbf{x}}$. In this section, we are interested in the existence and properties of $\nabla g_{\mathbf{x}}$.

PROPOSITION 4.1. *We suppose that $g_{\mathbf{x}}$ is Gaussian and that its auto-correlation function R is $\mathscr{C}^2(\mathbb{R}^d, \mathbb{R}^d)$. Then the following hold:*

- *$\nabla g_{\mathbf{x}}$ exists in the mean square sense, is Gaussian, and has $\nabla m_k(\mathbf{x})$ as mean and $\partial^2_{\mathbf{xy}} R(\mathbf{x}, \mathbf{x})$ as variance.*
- *The vector $[g_{\mathbf{x}}, \nabla g_{\mathbf{x}}]$ is Gaussian and has*

$$\begin{pmatrix} 1 & \partial_{\mathbf{y}} R(\mathbf{x}, \mathbf{x})^T \\ \partial_{\mathbf{y}} R(\mathbf{x}, \mathbf{x}) & \partial^2_{\mathbf{xy}} R(\mathbf{x}, \mathbf{x}) \end{pmatrix},$$

as variance.

Proof. As the auto-correlation function is $\mathscr{C}^2(\mathbb{R}^d, \mathbb{R}^d)$, the derivative of $g_{\mathbf{x}}$ exists (following [37, Appendix 9.A]). Moreover, as $g_{\mathbf{x}}$ is a Gaussian process, the vector $[g_{\mathbf{x}_1}, g_{\mathbf{x}_2}, \ldots, g_{\mathbf{x}_k}]$ is Gaussian for any $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}^d$. As a consequence, $\nabla g_{\mathbf{x}}$, but also $[g_{\mathbf{x}}, \nabla g_{\mathbf{x}}]$ are Gaussian vector, which are characterized by their first and second

order moments. The mean of $\nabla g_{\mathbf{x}}$ is $\nabla m_k(\mathbf{x})$, and for any $\mathbf{n}_1, \mathbf{n}_2 \in \mathbb{R}^d$,

$$\begin{aligned}\operatorname{Cov}\left(\frac{g_{\mathbf{x}+\tau_1\mathbf{n}_1}-g_{\mathbf{x}}}{\tau_1}, \frac{g_{\mathbf{x}+\tau_2\mathbf{n}_2}-g_{\mathbf{x}}}{\tau_2}\right) &= \frac{1}{\tau_1\tau_2}\left(R(\mathbf{x}+\tau_1\mathbf{n}_1, \mathbf{x}+\tau_2\mathbf{n}_2)\right.\\ &\quad -R(\mathbf{x}+\tau_1\mathbf{n}_1, \mathbf{x})\\ &\quad \left.-R(\mathbf{x}, \mathbf{x}+\tau_2\mathbf{n}_2)+R(\mathbf{x}, \mathbf{x})\right)\\ &\xrightarrow[\tau_1,\tau_2\to 0]{} \mathbf{n}_1^T\partial^2_{\mathbf{xy}}R(\mathbf{x}, \mathbf{x})\mathbf{n}_2.\end{aligned}$$

Thus, $\operatorname{Cov}(\nabla g_{\mathbf{x}}, \nabla g_{\mathbf{x}}) = \partial^2_{\mathbf{xy}}R(\mathbf{x}, \mathbf{x})$.

The mean of $[g_{\mathbf{x}}, \nabla g_{\mathbf{x}}]$ is $[m_k(\mathbf{x}), \nabla m_k(\mathbf{x})]$. For any $\mathbf{n} \in \mathbb{R}^d$, we have

$$\begin{aligned}\operatorname{Cov}\left(g_{\mathbf{x}}, \frac{g_{\mathbf{x}+\tau\mathbf{n}}-g_{\mathbf{x}}}{\tau}\right) &= \frac{R(\mathbf{x}, \mathbf{x}+\tau\mathbf{n})-R(\mathbf{x})}{\tau}\\ &\xrightarrow[\tau\to 0]{} \partial_{\mathbf{y}}R(\mathbf{x}, \mathbf{x})\cdot\mathbf{n},\end{aligned}$$

so that $\operatorname{Cov}(g_{\mathbf{x}}, \nabla g_{\mathbf{x}}) = \partial_{\mathbf{y}}R(\mathbf{x}, \mathbf{x})$. As a consequence, the covariance matrix of $[g_{\mathbf{x}}, \nabla g_{\mathbf{x}}]$ is

$$\begin{pmatrix} 1 & \partial_{\mathbf{y}}R(\mathbf{x}, \mathbf{x})^T \\ \partial_{\mathbf{y}}R(\mathbf{x}, \mathbf{x}) & \partial^2_{\mathbf{xy}}R(\mathbf{x}, \mathbf{x}) \end{pmatrix}.$$

□

4.2. Measurability of the process. For ensuring that (4.1) is measurable when $f = g_{\mathbf{x}}$, we must add some hypothesis on $g_{\mathbf{x}}$. This is summarized in the following proposition

PROPOSITION 4.2. *$\nabla\chi$ is measurable if and only if $\partial_{\mathbf{y}}R(\mathbf{x}, \mathbf{x}) = 0$ and $\partial^2_{\mathbf{xy}}R(\mathbf{x}, \mathbf{x})$ is nonnegative.*

Proof. Considering (4.1) with the random process $g_{\mathbf{x}}$ requires the measurability of

$$\delta_{\{x_1=0\}}\,\mathbf{x}_d\,\frac{1}{((2\pi)^{d+1}\det\Sigma)^{1/2}}\,\exp\left(-\frac{X^T\Sigma^{-1}X}{2}\right) \tag{4.2}$$

with

$$\Sigma = \begin{pmatrix} 1 & \partial_{\mathbf{y}}R(\mathbf{x}, \mathbf{x})^T \\ \partial_{\mathbf{y}}R(\mathbf{x}, \mathbf{x}) & \partial^2_{\mathbf{xy}}R(\mathbf{x}, \mathbf{x}) \end{pmatrix},$$

and $X = (x_1 - m_k(\mathbf{x}), \mathbf{x}_d - \nabla m_k(\mathbf{x}))$. Integrating in x_1 gives

$$\mathbf{x}_d\,\frac{1}{((2\pi)^{d+1}\det\Sigma)^{1/2}}\,\exp\left(-\frac{\check{X}^T\Sigma^{-1}\check{X}}{2}\right)$$

with $\check{X} = (-m_k(\mathbf{x}), \mathbf{x}_d - \nabla m_k(\mathbf{x}))$. The argument of the exponential can be developed as

$$\begin{aligned}\check{X}^T\Sigma\check{X} = &-m_k(\mathbf{x})^2 + 2\partial_{\mathbf{y}}R(\mathbf{x}, \mathbf{x})\cdot(\mathbf{x}_d - \nabla m_k(\mathbf{x}))\\ &+(\mathbf{x}_d - \nabla m_k(\mathbf{x}))^T\partial^2_{\mathbf{xy}}R(\mathbf{x}, \mathbf{x})(\mathbf{x}_d - \nabla m_k(\mathbf{x})).\end{aligned}$$

If we want to ensure that the argument of the exponential tends to $-\infty$ for any $\mathbf{x}_d$ tending to infinity, this requires that $\partial_{\mathbf{y}}R(\mathbf{x}, \mathbf{x}) = 0$ and that $\partial^2_{\mathbf{xy}}R(\mathbf{x}, \mathbf{x})$ is nonnegative.

Reciprocally, if $\partial_{\mathbf{y}} R(\mathbf{x},\mathbf{x}) = 0$ and $\partial^2_{\mathbf{xy}} R(\mathbf{x},\mathbf{x})$ is nonnegative, then

$$\delta_{\{x_1=0\}} \frac{\exp(-m_k(\mathbf{x})^2/2)}{((2\pi)^{d+1}\det \partial^2_{\mathbf{xy}} R(\mathbf{x},\mathbf{x}))^{1/2}} \mathrm{e}^{(\mathbf{x}_d - \nabla m_k(\mathbf{x}))^T \partial^2_{\mathbf{xy}} R(\mathbf{x},\mathbf{x})^{-1} (\mathbf{x}_d - \nabla m_k(\mathbf{x}))}$$

is measurable. □

Note that under the hypothesis of Proposition 4.2, we find $\mathbb{E}\left[\nabla \chi(\mathbf{x})\right] = \nabla \alpha_k(\mathbf{x})$. Indeed, denoting by

$$\check{\Sigma} = \partial^2_{\mathbf{xy}} R(\mathbf{x},\mathbf{x}),$$

we can compute

$$\begin{aligned}
\mathbb{E}\left[\nabla \chi_k(\mathbf{x})\right] &= \frac{\mathrm{e}^{-m_k(\mathbf{x})^2/2}}{((2\pi)^{d+1}\det \check{\Sigma})^{1/2}} \int_{x_1\in\mathbb{R}} \int_{\mathbf{x}_d\in\mathbb{R}^d} \delta_{\{x_1=0\}} \mathbf{x}_d \mathrm{e}^{-\check{X}^T \check{\Sigma}^{-1} \check{X}^T/2} \mathrm{d}x_1 \mathrm{d}\mathbf{x}_d \\
&= \frac{\mathrm{e}^{-m_k(\mathbf{x})^2/2}}{((2\pi)^{d+1}\det \check{\Sigma})^{1/2}} \int_{\mathbf{x}_d\in\mathbb{R}^d} \mathbf{x}_d \mathrm{e}^{-\check{X}^T \check{\Sigma}^{-1} \check{X}^T/2} \mathrm{d}\mathbf{x}_d \\
\mathbb{E}\left[\nabla \chi_k(\mathbf{x})\right] &= \frac{\mathrm{e}^{-m_k(\mathbf{x})^2/2}}{((2\pi)^{d+1}\det \check{\Sigma})^{1/2}} \int_{\mathbf{x}_d\in\mathbb{R}^d} \check{X} \mathrm{e}^{-\check{X}^T \check{\Sigma}^{-1} \check{X}^T/2} \mathrm{d}\mathbf{x}_d \\
&\quad + \frac{\mathrm{e}^{-m_k(\mathbf{x})^2/2} \nabla m_k(\mathbf{x})}{\sqrt{2\pi}} \frac{1}{((2\pi)^d \det \check{\Sigma})^{1/2}} \int_{\mathbf{x}_d\in\mathbb{R}^d} \mathrm{e}^{-\check{X}^T \check{\Sigma}^{-1} \check{X}^T/2} \mathrm{d}\mathbf{x}_d.
\end{aligned}$$

In this last formula, the first integral vanishes, whereas the Gaussian law can be recognized in the second one, which gives

$$\mathbb{E}\left[\nabla \chi_k(\mathbf{x})\right] = \frac{\mathrm{e}^{-m_k(\mathbf{x})^2/2} \nabla m_k(\mathbf{x})}{\sqrt{2\pi}} = \nabla \alpha_k(\mathbf{x}).$$

Once we have been able to define the context of measurability of (2.9) when f is a Gaussian process, we can now state that with our Gaussian model, we have

$$\begin{aligned}
&\mathbb{E}\left[\hat{\mathbf{F}}^{lag}_{\bar{k}k}\left(\frac{\nabla f(\mathbf{x})}{\|\nabla f(\mathbf{x})\|}\right)\|\nabla f(\mathbf{x})\|\right] \\
&= \frac{\exp(-m_k(\mathbf{x})^2/2)}{((2\pi)^{d+1}\det \partial^2_{\mathbf{xy}} R(\mathbf{x},\mathbf{x}))^{1/2}} \\
&\quad \times \int_{x_1\in\mathbb{R},\mathbf{x}_d\in\mathbb{R}^d} \delta_{\{x_1=0\}} \mathrm{e}^{\frac{(\mathbf{x}_d - \nabla m_k(\mathbf{x}))^T \partial^2_{\mathbf{xy}} R(\mathbf{x},\mathbf{x})^{-1} (\mathbf{x}_d - \nabla m_k(\mathbf{x}))}{2}} \hat{\mathbf{F}}^{lag}_{\bar{k}k}\left(\frac{\mathbf{x}_d}{\|\mathbf{x}_d\|}\right)\|\mathbf{x}_d\|.
\end{aligned}$$

This integral can be integrated once in x_1, and if we denote by $\check{\Sigma} = \partial^2_{\mathbf{xy}} R(\mathbf{x},\mathbf{x})^{-1}$, we get

$$\begin{aligned}
&\mathbb{E}\left[\hat{\mathbf{F}}^{lag}_{\bar{k}k}\left(\frac{\nabla f(\mathbf{x})}{\|\nabla f(\mathbf{x})\|}\right)\|\nabla f(\mathbf{x})\|\right] \\
&= \frac{\exp(-m_k(\mathbf{x})^2/2)}{((2\pi)^{d+1}\det \check{\Sigma})^{1/2}} \\
&\quad \times \int_{\mathbf{x}_d\in\mathbb{R}^d} \mathrm{e}^{\frac{(\mathbf{x}_d - \nabla m_k(\mathbf{x}))^T \check{\Sigma}^{-1} (\mathbf{x}_d - \nabla m_k(\mathbf{x}))}{2}} \hat{\mathbf{F}}^{lag}_{\bar{k}k}\left(\frac{\mathbf{x}_d}{\|\mathbf{x}_d\|}\right)\|\mathbf{x}_d\| \, \mathrm{d}\mathbf{x}_d.
\end{aligned} \tag{4.3}$$

5. Some particular cases. In this section, particular cases are studied, in which the integral found in (4.3) can be either exactly computed or approximated. The different cases rely on the following criteria:

- the *memory* of the stochastic process, more precisely the relative behavior of $\nabla m_k(\mathbf{x})$ and $\check{\Sigma}$,
- the *tropicity* of the self-correlation of the stochastic process,
- the *relative state* of the two fluids, which may induce a simplification in the Lagrangian flux that is averaged.

5.1. Local contact. When locally the two fluids are at mechanical equilibrium, (4.3) can be computed explicitly.

PROPOSITION 5.1. *If for a given* $\mathbf{x}$ *we have* $\mathbf{u}_k(\mathbf{x}) = \mathbf{u}_{\bar{k}}(\mathbf{x}) = \mathbf{u}_0$ *and* $P_k(\mathbf{x}) = P_{\bar{k}}(\mathbf{x}) = P_0$*, then* (4.3) *gives* (1.1) *with* $P_I = P_0$ *and* $\mathbf{u}_I = \mathbf{u}_0$ *(and with undetermined* λ *and* μ*, as these terms are not active in this case).*

Proof. When locally, the phases have the same velocity $\mathbf{u}_0$ and pressure P_0, the solution of the Riemann problem in the direction $\mathbf{n}$ is a contact with velocity $\mathbf{u}_0 \cdot \mathbf{n}$ and pressure P_0. Then, for all $\mathbf{n}$, $P^\star(\mathbf{n}) = P_0$ and $\mathbf{u}^\star(\mathbf{n}) = \mathbf{u}_0 \cdot \mathbf{n}$, so that

$$\hat{\mathbf{F}}^{lag}(\mathbf{x}_d)\,\|\mathbf{x}_d\| = \begin{pmatrix} -\mathbf{u}_0 \cdot \dfrac{\mathbf{x}_d}{\|\mathbf{x}_d\|} \\ 0 \\ P_0 \dfrac{\mathbf{x}_d}{\|\mathbf{x}_d\|} \\ P_0 \mathbf{u}_0 \cdot \dfrac{\mathbf{x}_d}{\|\mathbf{x}_d\|} \end{pmatrix} \|\mathbf{x}_d\| = \begin{pmatrix} -\mathbf{u}_0 \cdot \mathbf{x}_d \\ 0 \\ P_0\, \mathbf{x}_d \\ P_0 \mathbf{u}_0 \cdot \mathbf{x}_d \end{pmatrix}.$$

In the integral, it remains the expectancy of $\mathbf{x}_d$, which gives

$$\mathbb{E}\left[(\hat{\mathbf{F}}_k(\hat{\mathbf{U}}_k) - \mathbf{v}_i \hat{\mathbf{U}}_k)\nabla\chi_k\right] = \begin{pmatrix} -\mathbf{u}_0 \cdot \nabla\alpha_k \\ 0 \\ P_0 \nabla\alpha_k \\ P_0 \mathbf{u}_0 \cdot \nabla\alpha_k \end{pmatrix}.$$

This gives the model (1.1) with $P_I = P_0$ and $\mathbf{u}_I = \mathbf{u}_0$. ∎

5.2. Long memory case. The long memory case matches with the case in which the auto-correlation function R is flat, more precisely, when $\partial^2_{\mathbf{xy}} R(\mathbf{x}, \mathbf{x}) = 0$. In this case, the following hold.

PROPOSITION 5.2 (Long memory case). *When* $\partial^2_{\mathbf{xy}} R(\mathbf{x}, \mathbf{x}) \to 0$, (4.3) *gives the system* (1.1) *with* $\lambda = \mu = 0$*, and the following interfacial velocity and pressure:*

$$\mathbf{u}_I = u^\star\left(\frac{\nabla\alpha(\mathbf{x})}{\|\nabla\alpha(\mathbf{x})\|}\right)\frac{\nabla\alpha(\mathbf{x})}{\|\nabla\alpha(\mathbf{x})\|} \quad \textit{and} \quad P_I = P^\star\left(\frac{\nabla\alpha(\mathbf{x})}{\|\nabla\alpha(\mathbf{x})\|}\right).$$

Proof. the integral of the function

$$\frac{1}{((2\pi)^d \det\check{\Sigma})^{1/2}}\, \mathrm{e}^{\left(-\frac{(\mathbf{x}_d - \nabla m)^T \check{\Sigma}^{-1}(\mathbf{x}_d - \nabla m)}{2}\right)}$$

over $\mathbb{R}^d$ tends to an approximation of the identity concentrated in $\nabla m_k(\mathbf{x})$, which gives

$$\mathbb{E}\left[(\hat{\mathbf{F}}_k(\hat{\mathbf{U}}_k) - \mathbf{v}_i \hat{\mathbf{U}}_k)\nabla\chi_k\right] = \frac{\exp(-m_k(\mathbf{x})^2/2)}{\sqrt{2\pi}}\, \hat{\mathbf{F}}^{lag}_{\bar{k}k}\left(\frac{\nabla m_k(\mathbf{x})}{\|\nabla m_k(\mathbf{x})\|}\right)\|\nabla m_k(\mathbf{x})\|.$$

As $\nabla m_k(\mathbf{x})$ and $\nabla\alpha_k(\mathbf{x})$ are colinear, we may rewrite

$$u^\star\left(\frac{\nabla m_k(\mathbf{x})}{\|\nabla m_k(\mathbf{x})\|}\right) = u^\star\left(\frac{\nabla\alpha_k(\mathbf{x})}{\|\nabla\alpha_k(\mathbf{x})\|}\right)\frac{\nabla\alpha_k(\mathbf{x})}{\|\nabla\alpha_k(\mathbf{x})\|} \cdot \frac{\nabla m_k(\mathbf{x})}{\|\nabla m_k(\mathbf{x})\|}.$$

Setting

$$\mathbf{u}_I = u^\star\left(\frac{\nabla\alpha_k(\mathbf{x})}{\|\nabla\alpha_k(\mathbf{x})\|}\right)\frac{\nabla\alpha_k(\mathbf{x})}{\|\nabla\alpha_k(\mathbf{x})\|} \quad \text{and} \quad P_I = P^\star\left(\frac{\nabla\alpha_k(\mathbf{x})}{\|\nabla\alpha_k(\mathbf{x})\|}\right),$$

we find

$$\begin{aligned}\mathbb{E}\left[(\hat{\mathbf{F}}_k(\hat{\mathbf{U}}_k) - \mathbf{v}_i\hat{\mathbf{U}}_k)\nabla\chi_k\right] &= \frac{\exp(-m_k(\mathbf{x})^2/2)}{\sqrt{2\pi}}\begin{pmatrix} -\mathbf{u}_I\cdot\nabla m_k(\mathbf{x}) \\ 0 \\ P_I\nabla m_k(\mathbf{x}) \\ P_I\mathbf{u}_I\cdot\nabla m_k(\mathbf{x})\end{pmatrix} \\ &= \begin{pmatrix} -\mathbf{u}_I\cdot\nabla\alpha_k(\mathbf{x}) \\ 0 \\ P_I\nabla\alpha_k(\mathbf{x}) \\ P_I\mathbf{u}_I\cdot\nabla\alpha_k(\mathbf{x})\end{pmatrix}.\end{aligned}$$

□

Note that this case was previously derived in [17]. It also matches with the limit model of the numerical method described in [2] and matches in one dimension with the closure derived in [8].

5.3. One dimensional case. In the one dimensional case, the auto-correlation matrix $\check{\Sigma}$ is a nonnegative scalar which we denote by λ_1^2. Then (4.3) can be computed explicitly and gives the system (1.2).

PROPOSITION 5.3 (One dimensional case). *In one dimension, setting*

$$w := \operatorname{erf}\left(\frac{|\partial_x m_k(x)|}{\lambda_1\sqrt{2}}\right),$$

the system is closed by the following interfacial quantities

$$\begin{cases} \mathbf{u}_I = \operatorname{sgn}(\partial_x\alpha_k(x))\left(\dfrac{1+w}{2}\,u^\star_{\bar{k}k}(\partial_x\alpha_k(x)) + \dfrac{w-1}{2}\,u^\star_{\bar{k}k}(-\partial_x\alpha_k(x))\right), \\ P_I = \operatorname{sgn}(\partial_x\alpha_k(x))\left(\dfrac{1+w}{2}\,P^\star_{\bar{k}k}(\partial_x\alpha_k(x)) + \dfrac{w-1}{2}\,P^\star_{\bar{k}k}(-\partial_x\alpha_k(x))\right), \\ (P\mathbf{u})_I = \operatorname{sgn}(\partial_x\alpha_k(x))\left(\dfrac{1+w}{2}\,P^\star_{\bar{k}k}(\partial_x\alpha_k(x))u^\star_{\bar{k}k}(\partial_x\alpha_k(x))\right. \\ \qquad\qquad \left. + \dfrac{w-1}{2}\,P^\star_{\bar{k}k}(-\partial_x\alpha_k(x))u^\star_{\bar{k}k}(-\partial_x\alpha_k(x))\right), \end{cases},$$

and includes the nonlinear relaxation term

$$\lambda_1 \mathrm{e}^{-\frac{\partial_x m_k(x)^2}{2\lambda_1^2}}\frac{\mathrm{e}^{-\frac{m_k(x)^2}{2}}}{2\pi}\begin{pmatrix} u^\star_{\bar{k}k}(-\partial_x\alpha(x)) - u^\star_{\bar{k}k}(\partial_x\alpha(x)) \\ 0 \\ P^\star_{\bar{k}k}(\partial_x\alpha(x)) - P^\star_{\bar{k}k}(-\partial_x\alpha(x)) \\ P^\star_{\bar{k}k}(\partial_x\alpha(x))u^\star_{\bar{k}k}(\partial_x\alpha(x)) - P^\star_{\bar{k}k}(-\partial_x\alpha(x))u^\star_{\bar{k}k}(-\partial_x\alpha(x))\end{pmatrix}.$$

Proof. We are interested in the computation of

$$\frac{\mathrm{e}^{-m_k(x)^2/2}}{\sqrt{2\pi}}\frac{1}{\lambda_1\sqrt{2\pi}}\int_{x'\in\mathbb{R}}\mathrm{e}^{-(x'-\partial_x m_k(x))^2/(2\lambda_1^2)}\hat{\mathbf{F}}^{lag}_{\bar{k}k}(x')\,|x'|\,\mathrm{d}x.$$

In the integral, we change the variable x' by u defined as

$$u = \frac{x' - \partial_x m_k(x)}{\lambda_1},$$

and this gives

$$\frac{\mathrm{e}^{-m_k(x)^2/2}}{\sqrt{2\pi}}\,\frac{1}{\lambda_1\sqrt{2\pi}}\int_{u\in\mathbb{R}}\mathrm{e}^{-u^2/2}\hat{\mathbf{F}}^{lag}_{\bar{k}k}\left(\partial_x m_k(x)+\lambda_1 u\right)\left|\partial_x m_k(x)+\lambda_1 u\right|\,\lambda_1\mathrm{d}u,$$

which can be simplified as

$$\frac{\mathrm{e}^{-m_k(x)^2/2}}{2\pi}\int_{u\in\mathbb{R}}\mathrm{e}^{-u^2/2}\hat{\mathbf{F}}^{lag}_{\bar{k}k}\left(\partial_x m_k(x)+\lambda_1 u\right)\left|\partial_x m_k(x)+\lambda_1 u\right|\,\mathrm{d}u. \tag{5.1}$$

We remark that $\hat{\mathbf{F}}^{lag}_{\bar{k}k}\left(\partial_x m_k(x)+\lambda_1 u\right)$ may take two values, depending on whether it is positive or negative. We therefore divide the integration domain of u into two domains: $u\le -\partial_x m_k(x)/\lambda_1$ and $u\ge -\partial_x m_k(x)/\lambda_1$,

$$\begin{aligned}
&\int_{u\in\mathbb{R}}\mathrm{e}^{-u^2/2}\hat{\mathbf{F}}^{lag}_{\bar{k}k}\left(\partial_x m_k(x)+\lambda_1 u\right)\left|\partial_x m_k(x)+\lambda_1 u\right|\,\mathrm{d}u\\
&\quad=-\int_{-\infty}^{-\partial_x m_k(x)/\lambda_1}\mathrm{e}^{-u^2/2}\hat{\mathbf{F}}^{lag}_{\bar{k}k}\left(-1\right)\left(\partial_x m_k(x)+\lambda_1 u\right)\,\mathrm{d}u\\
&\qquad+\int_{-\partial_x m_k(x)/\lambda_1}^{\infty}\mathrm{e}^{-u^2/2}\hat{\mathbf{F}}^{lag}_{\bar{k}k}\left(1\right)\left(\partial_x m_k(x)+\lambda_1 u\right)\,\mathrm{d}u.
\end{aligned}$$

We gather on one hand the terms including $\partial_x m_k(x)$, and on the other hand the ones factorized by λ_1

- **Terms including $\partial_x m_k(x)$.** The terms including $\partial_x m_k(x)$ are

$$\begin{aligned}
&\partial_x m_k(x)\Bigg\{-\hat{\mathbf{F}}^{lag}_{\bar{k}k}(-1)\int_{-\infty}^{-\partial_x m_k(x)/\lambda_1}\mathrm{e}^{-u^2/2}\,\mathrm{d}u\\
&\qquad+\hat{\mathbf{F}}^{lag}_{\bar{k}k}(1)\int_{-\partial_x m_k(x)/\lambda_1}^{\infty}\mathrm{e}^{-u^2/2}\,\mathrm{d}u\Bigg\}\\
&=\partial_x m_k(x)\sqrt{\frac{\pi}{2}}\Bigg\{-\hat{\mathbf{F}}^{lag}_{\bar{k}k}(-1)\left(\mathrm{erf}\left(\frac{-\partial_x m_k(x)}{\lambda_1\sqrt{2}}\right)+1\right)\\
&\qquad+\hat{\mathbf{F}}^{lag}_{\bar{k}k}(1)\left(1-\mathrm{erf}\left(\frac{-\partial_x m_k(x)}{\lambda_1\sqrt{2}}\right)\right)\Bigg\}\\
&=\partial_x m_k(x)\sqrt{\frac{\pi}{2}}\Bigg\{\hat{\mathbf{F}}^{lag}_{\bar{k}k}(1)-\hat{\mathbf{F}}^{lag}_{\bar{k}k}(-1)\\
&\qquad-(\hat{\mathbf{F}}^{lag}_{\bar{k}k}(1)+\hat{\mathbf{F}}^{lag}_{\bar{k}k}(-1))\mathrm{erf}\left(\frac{-\partial_x m_k(x)}{\lambda_1\sqrt{2}}\right)\Bigg\}.
\end{aligned}$$

  In the other closures found, the direction $\nabla\alpha_k/\left\|\nabla\alpha_k\right\|$ appeared as a factor and as a direction for the Riemann problems. In one dimension, this term is equal to $\mathrm{sgn}(\partial_x\alpha_k(x))$, or equivalently $\mathrm{sgn}(\partial_x m_k(x))$. Then the term may be rewritten

$$\begin{aligned}
&\partial_x m_k(x)\mathrm{sgn}(\partial_x\alpha_k(x))\sqrt{\frac{\pi}{2}}\Bigg\{\hat{\mathbf{F}}^{lag}_{\bar{k}k}(\partial_x\alpha_k(x))-\hat{\mathbf{F}}^{lag}_{\bar{k}k}(-\partial_x\alpha_k(x))\\
&\qquad+(\hat{\mathbf{F}}^{lag}_{\bar{k}k}(\partial_x\alpha_k(x))+\hat{\mathbf{F}}^{lag}_{\bar{k}k}(-\partial_x\alpha_k(x)))\mathrm{erf}\left(\frac{|\partial_x m_k(x)|}{\lambda_1\sqrt{2}}\right)\Bigg\}.
\end{aligned}$$

Using the factor coming from (5.1) gives

$$(\partial_x \alpha_k(x))\mathrm{sgn}(\partial_x \alpha_k(x)) \left(\frac{1+w}{2} \hat{\mathbf{F}}^{lag}_{\bar{k}k}(\partial_x \alpha_k(x)) + \frac{w-1}{2} \hat{\mathbf{F}}^{lag}_{\bar{k}k}(-\partial_x \alpha_k(x)) \right).$$

- **Terms factorized by λ_1.** Terms factorized by λ_1 are

$$\begin{aligned}
&\lambda_1 \Bigg\{ - \hat{\mathbf{F}}^{lag}_{\bar{k}k}(-1) \int_{-\infty}^{-\partial_x m_k(x)/\lambda_1} u \mathrm{e}^{-u^2/2} \, \mathrm{d}u \\
&\qquad + \hat{\mathbf{F}}^{lag}_{\bar{k}k}(1) \int_{-\partial_x m_k(x)/\lambda_1}^{\infty} u \mathrm{e}^{-u^2/2} \, \mathrm{d}u \Bigg\} \\
&= \lambda_1 \Bigg\{ - \hat{\mathbf{F}}^{lag}_{\bar{k}k}(-1) [-\mathrm{e}^{-u^2/2}]_{-\infty}^{-\partial_x m_k(x)/\lambda_1} \\
&\qquad + \hat{\mathbf{F}}^{lag}_{\bar{k}k}(1) [-\mathrm{e}^{-u^2/2}]_{-\partial_x m_k(x)/\lambda_1}^{\infty} \Bigg\} \\
&= \lambda_1 \mathrm{e}^{-(\partial_x m_k(x))^2/2\lambda_1^2} \left(\hat{\mathbf{F}}^{lag}_{\bar{k}k}(\partial_x \alpha_k(x)) + \hat{\mathbf{F}}^{lag}_{\bar{k}k}(-\partial_x \alpha_k(x)) \right).
\end{aligned}$$

□

We remark that in one dimension, if $\lambda_1 \to 0$, then $w \to 1$, and the closure of Proposition 5.2 is recovered.

5.4. Short memory case. The modeling we are using essentially includes two length scales: one is given by the invert of the spatial derivative of the volume fraction, $\nabla \alpha_k(\mathbf{x})$, and another one is given by the invert of the square root of the eigenvalues of the self-correlation function. If we suppose that the stochastic process has a short memory compared with the variations of α_k, then we are in a short memory case, in which an asymptotic expansion of (4.3) can clearly make appear a relaxation and an interfacial term.

PROPOSITION 5.4 (Short memory case). *We denote by Λ^2 a diagonalization of $\check{\Sigma}$ and Q an orthogonal matrix such that $\check{\Sigma} = Q^T \Lambda^2 Q$. Also, we denote by $|\mathbb{V}_d|$ the volume of the d-dimensional unit ball, and by $\left|\mathbb{S}^{d-1}\right|$ the surface of the unit sphere in dimension d.*

If we suppose that $\Lambda^{-1} \nabla m_k(\mathbf{x}) \ll 1$ then the system (1.2) is found with the following closure of the interfacial terms

$$\left\{ \begin{aligned}
\mathbf{u}_I &= \frac{\check{\Sigma}^{-1}}{|\mathbb{V}_d| \left|\det \check{\Sigma}\right|^{1/2}} \int_{\mathbf{x}_d \in Q^T \Lambda(\mathbb{S}^{d-1})} u^{\star}_{\bar{k}k} \left(\frac{\mathbf{x}_d}{\|\mathbf{x}_d\|} \right) \|\mathbf{x}_d\| \, \mathbf{x}_d \mathrm{d}\mathbf{x}_d, \\
\mathbf{P}_I &= \frac{1}{|\mathbb{V}_d| \left|\det \check{\Sigma}\right|^{1/2}} \left(\int_{\mathbf{x}_d \in Q^T \Lambda(\mathbb{S}^{d-1})} P^{\star}_{\bar{k}k} \left(\frac{\mathbf{x}_d}{\|\mathbf{x}_d\|} \right) \mathbf{x}_d \mathbf{x}_d^T \mathrm{d}\mathbf{x}_d \right) \check{\Sigma}^{-1}, \\
(P\mathbf{u})_I &= \frac{\check{\Sigma}^{-1}}{|\mathbb{V}_d| \left|\det \check{\Sigma}\right|^{1/2}} \int_{\mathbf{x}_d \in Q^T \Lambda(\mathbb{S}^{d-1})} (Pu)^{\star}_{\bar{k}k} \left(\frac{\mathbf{x}_d}{\|\mathbf{x}_d\|} \right) \|\mathbf{x}_d\| \, \mathbf{x}_d \mathrm{d}\mathbf{x}_d,
\end{aligned} \right.$$

and with the following nonlinear relaxation terms,

$$\left\{\begin{array}{l} \mathcal{R}_k^{(\alpha)} = \dfrac{\exp(-m_k(\mathbf{x})^2/2)}{2\pi\,|\mathbb{V}_{d-1}|\,\left|\det\check{\Sigma}\right|^{1/2}} \displaystyle\int_{\mathbf{x}_d\in Q^T\Lambda(\mathbb{S}^{d-1})} (-u_{\bar{k}k}^\star)\left(\frac{\mathbf{x}_d}{\|\mathbf{x}_d\|}\right)\|\mathbf{x}_d\|\,\mathrm{d}\mathbf{x}_d, \\ \mathcal{R}_k^{(\rho\mathbf{u})} = \dfrac{\exp(-m_k(\mathbf{x})^2/2)}{2\pi\,|\mathbb{V}_{d-1}|\,\left|\det\check{\Sigma}\right|^{1/2}} \displaystyle\int_{\mathbf{x}_d\in Q^T\Lambda(\mathbb{S}^{d-1})} P_{\bar{k}k}^\star\left(\frac{\mathbf{x}_d}{\|\mathbf{x}_d\|}\right)\mathbf{x}_d\mathrm{d}\mathbf{x}_d, \\ \mathcal{R}_k^{(\rho E)} = \dfrac{\exp(-m_k(\mathbf{x})^2/2)}{2\pi\,|\mathbb{V}_{d-1}|\,\left|\det\check{\Sigma}\right|^{1/2}} \displaystyle\int_{\mathbf{x}_d\in Q^T\Lambda(\mathbb{S}^{d-1})} (Pu)_{\bar{k}k}^\star\left(\frac{\mathbf{x}_d}{\|\mathbf{x}_d\|}\right)\|\mathbf{x}_d\|\,\mathrm{d}\mathbf{x}_d. \end{array}\right.$$

Note that the relaxation terms look rather as relaxation terms if the integral on the sphere $\mathbb{S}^{d-1}$ is done on a semisphere, by assembling the terms of opposite direction.

Proof. With the hypothesis we have, the integral we have to evaluate is equal to

$$\frac{\exp\left(-\frac{m_k(\mathbf{x})^2}{2}\right)}{\sqrt{2\pi}}\,\frac{1}{(2\pi)^{d/2}(\det\check{\Sigma})^{1/2}}\int_{\mathbf{x}_d\in\mathbb{R}^d}\mathrm{e}^{\left(-\frac{(\mathbf{x}_d-\nabla m_k)^T\check{\Sigma}^{-1}(\mathbf{x}_d-\nabla m_k)}{2}\right)}\hat{\mathbf{F}}_{\bar{k}k}^{lag}(\mathbf{x}_d)\,\|\mathbf{x}_d\|\,\mathrm{d}\mathbf{x}_d.$$

The existence of Q and Λ^2 is ensured because $\check{\Sigma}$ is strictly nonnegative. It can therefore be diagonalized in an orthonormal basis. Denoting Q the transformation matrix toward an orthonormal basis of diagonalization of $\check{\Sigma}$, and Λ the diagonal matrix with the square roots of the eigenvalues of $\check{\Sigma}$, we have $\check{\Sigma} = Q^T\Lambda^2 Q$. We perform the variable change $\mathbf{y}_d = \Lambda^{-1}Q\mathbf{x}_d$. As Q is the matrix of an isometry, $\|\mathbf{x}_d\| = \left\|Q^T\Lambda\mathbf{y}_d\right\| = \|\Lambda\mathbf{y}_d\|$. Also, as $\det Q = 1$, the determinant of the variable change is the product of the square roots of the eigenvalues of $\check{\Sigma}$, and so is equal to $(\det\check{\Sigma})^{1/2}$. This gives

$$\frac{\exp(-m_k(\mathbf{x})^2/2)}{\sqrt{2\pi}}\,\frac{1}{(2\pi)^{d/2}}\int_{\mathbf{y}_d\in\mathbb{R}^d}\mathrm{e}^{\left(-\frac{\left\|\mathbf{y}_d-\Lambda^{-1}Q\nabla m_k\right\|^2}{2}\right)}\hat{\mathbf{F}}_{\bar{k}k}^{lag}(Q^T\Lambda\mathbf{y}_d)\,\left\|Q^T\Lambda\mathbf{y}_d\right\|\,\mathrm{d}\mathbf{y}_d.$$

Developing the exponential gives

$$\begin{aligned}&\frac{\exp(-m_k(\mathbf{x})^2/2)}{\sqrt{2\pi}}\,\frac{\mathrm{e}^{\left(-\frac{\left\|\Lambda^{-1}Q\nabla m_k\right\|^2}{2}\right)}}{(2\pi)^{d/2}}\\ &\qquad\times\int_{\mathbf{y}_d\in\mathbb{R}^d}\mathrm{e}^{-\|\mathbf{y}_d\|^2/2}\mathrm{e}^{\left(\nabla m_k^T Q^T\Lambda^{-1}\mathbf{y}_d\right)}\hat{\mathbf{F}}_{\bar{k}k}^{lag}(Q^T\Lambda\mathbf{y}_d)\,\|\Lambda\mathbf{y}_d\|\,\mathrm{d}\mathbf{y}_d.\end{aligned}$$

If all the eigenvalues are of the same order, and large with respect to a given parameter λ, then

$$\begin{aligned}\mathrm{e}^{\left(-\frac{\left\|\Lambda^{-1}Q\nabla m_k\right\|^2}{2}\right)} &= 1+\mathcal{O}\left(\frac{1}{\lambda^2}\right)\\ \mathrm{e}^{\left(\nabla m_k^T Q^T\Lambda^{-1}\mathbf{y}_d\right)} &= 1+\nabla m_k^T Q^T\Lambda^{-1}\mathbf{y}_d+\mathcal{O}\left(\frac{1}{\lambda^2}\right).\end{aligned}\tag{5.2}$$

We may therefore identify the following.

- **The interfacial terms**

$$\begin{aligned}&\frac{\exp(-m_k(\mathbf{x})^2/2)}{\sqrt{2\pi}}\,\frac{1}{(2\pi)^{d/2}}\\ &\qquad\times\int_{\mathbf{y}_d\in\mathbb{R}^d}\mathrm{e}^{-\|\mathbf{y}_d\|^2/2}\left(\nabla m_k^T Q^T\Lambda^{-1}\mathbf{y}_d\right)\hat{\mathbf{F}}_{\bar{k}k}^{lag}(Q^T\Lambda\mathbf{y}_d)\,\|\Lambda\mathbf{y}_d\|\,\mathrm{d}\mathbf{y}_d,\end{aligned}$$

which can be rewritten

$$\frac{\nabla\alpha^T Q^T \Lambda^{-1}}{(2\pi)^{d/2}} \int_{\mathbf{y}_d \in \mathbb{R}^d} \mathrm{e}^{-\|\mathbf{y}_d\|^2/2} \mathbf{y}_d \hat{\mathbf{F}}^{lag}_{\bar{k}k}(Q^T \Lambda \mathbf{y}_d) \, \|\Lambda \mathbf{y}_d\| \, \mathrm{d}\mathbf{y}_d.$$

We remark that $\hat{\mathbf{F}}^{lag}_{\bar{k}k}(Q^T \Lambda \mathbf{y}_d)$ is homogeneous of degree 0 and that $\|\Lambda \mathbf{y}_d\|$ is homogeneous of degree 1. The integration can be made with a polar or spherical variable change, which can be written

$$\begin{aligned}
&\int_{\mathbf{y}_d \in \mathbb{R}^d} \mathrm{e}^{-\|\mathbf{y}_d\|^2/2} \mathbf{y}_d \hat{\mathbf{F}}^{lag}_{\bar{k}k}(Q^T \Lambda \mathbf{y}_d) \, \|\Lambda \mathbf{y}_d\| \, \mathrm{d}\mathbf{y}_d \\
&\quad = \int_0^\infty \mathrm{e}^{-r^2/2} r^{d+1} \, \mathrm{d}r \\
&\qquad \times \int_{\mathbf{y}_d \in \mathbb{S}^{d-1}} \mathbf{y}_d \hat{\mathbf{F}}^{lag}_{\bar{k}k}(Q^T \Lambda \mathbf{y}_d) \, \|\Lambda \mathbf{y}_d\| \, \mathrm{d}\mathbf{y}_d,
\end{aligned}$$

where $\mathbb{S}^{d-1}$ is the $(d-1)$ unit sphere. We recall that

$$\int_0^\infty r^n \mathrm{e}^{-r^2/2} \, \mathrm{d}r = 2^{\frac{n-1}{2}} \Gamma\left(\frac{n+1}{2}\right)$$

and that the volume of the unit ball in $\mathbb{R}^d$ is $|\mathbb{V}_d| = \dfrac{\pi^{d/2}}{\Gamma\left(\dfrac{d}{2}+1\right)}$, which gives the following compact formulation for the interfacial terms:

$$\nabla\alpha^T \frac{1}{|\mathbb{V}_d|} \int_{\mathbf{y}_d \in \mathbb{S}^{d-1}} Q^T \Lambda^{-1} \mathbf{y}_d \hat{\mathbf{F}}^{lag}_{\bar{k}k}(Q^T \Lambda \mathbf{y}_d) \, \|\Lambda \mathbf{y}_d\| \, \mathrm{d}\mathbf{y}_d.$$

Going back to the original variable for the integral $\mathbf{x}_d = Q^T \Lambda \mathbf{y}_d$ gives

$$\nabla\alpha^T \frac{\check{\Sigma}^{-1}}{|\mathbb{V}_d| \left|\det \check{\Sigma}\right|^{1/2}} \int_{\mathbf{x}_d \in Q^T \Lambda(\mathbb{S}^{d-1})} \mathbf{x}_d \hat{\mathbf{F}}^{lag}_{\bar{k}k}(\mathbf{x}_d) \, \|\mathbf{x}_d\| \, \mathrm{d}\mathbf{x}_d. \tag{5.3}$$

Note that in (5.3), the multiplication $\nabla\alpha^T \check{\Sigma}^{-1} \mathbf{x}_d$ is scalar. For the integration of the interfacial pressure term, for which $\hat{\mathbf{F}}^{lag}_{\bar{k}k}(\mathbf{x}_d)$ is a vector, it is more suited to move this scalar to the right of the integral and to transpose it.

- **The relaxation terms**

$$\frac{\exp(-m_k(\mathbf{x})^2/2)}{\sqrt{2\pi}} \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{y}_d \in \mathbb{R}^d} \mathrm{e}^{-\|\mathbf{y}_d\|^2/2} \hat{\mathbf{F}}^{lag}_{\bar{k}k}(Q^T \Lambda \mathbf{y}_d) \, \|\Lambda \mathbf{y}_d\| \, \mathrm{d}\mathbf{y}_d.$$

Remarks on the homogeneity of the different terms still holds, and the integral can be computed by

$$\begin{aligned}
&\int_{\mathbf{y}_d \in \mathbb{R}^d} \mathrm{e}^{-\|\mathbf{y}_d\|^2/2} \hat{\mathbf{F}}^{lag}_{\bar{k}k}(Q^T \Lambda \mathbf{y}_d) \, \|\Lambda \mathbf{y}_d\| \, \mathrm{d}\mathbf{y}_d \\
&\quad = \int_0^\infty \mathrm{e}^{-r^2/2} r^d \, \mathrm{d}r \\
&\qquad \times \int_{\mathbf{y}_d \in \mathbb{S}^{d-1}} \hat{\mathbf{F}}^{lag}_{\bar{k}k}(Q^T \Lambda \mathbf{y}_d) \, \|\Lambda \mathbf{y}_d\| \, \mathrm{d}\mathbf{y}_d \\
&\quad = 2^{\frac{d-1}{2}} \Gamma\left(\frac{d+1}{2}\right) \int_{\mathbf{y}_d \in \mathbb{S}^{d-1}} \hat{\mathbf{F}}^{lag}_{\bar{k}k}(Q^T \Lambda \mathbf{y}_d) \, \|\Lambda \mathbf{y}_d\| \, \mathrm{d}\mathbf{y}_d,
\end{aligned}$$

and we remark that

$$\frac{\exp(-m_k(\mathbf{x})^2/2)}{\sqrt{2\pi}}\,\frac{1}{(2\pi)^{d/2}}\,2^{\frac{d-1}{2}}\Gamma\left(\frac{d+1}{2}\right)=\frac{\exp(-m_k(\mathbf{x})^2/2)}{2\pi\,|\mathbb{V}_{d-1}|},$$

so that we finally find for the relaxation terms

$$\frac{\exp(-m_k(\mathbf{x})^2/2)}{2\pi\,|\mathbb{V}_{d-1}|\,\left|\det\check{\Sigma}\right|^{1/2}}\int_{\mathbf{x}_d\in Q^T\Lambda(\mathbb{S}^{d-1})}\hat{\mathbf{F}}^{lag}_{\bar{k}k}(\mathbf{x}_d)\,\|\mathbf{x}_d\|\,\mathrm{d}\mathbf{x}_d.$$

□

5.5. Short memory, linearized case. When the pressure and velocity of the two fluids are close, the expression of the Lagrangian flux can be linearized by using the acoustic approximation. In general, the solution of the Riemann problem in the normalized direction $\mathbf{n}_{\bar{k}k}$ is computed by intersecting the left and right *wave curves* in the $(P,\mathbf{u}\cdot\mathbf{n})$ plane, see [22, Chap II.3]. The Riemann problem can be formalized as follows:

$$\text{Find } \left(u^\star_{\bar{k}k},P^\star_{\bar{k}k}\right)\quad\text{such that}\qquad\begin{cases}u^\star_{\bar{k}k}=\mathbf{u}_{\bar{k}}\cdot\mathbf{n}_{\bar{k}k}-\mathscr{W}_{\bar{k}}\left(\mathbf{U}_{\bar{k}},P^\star_{\bar{k}k}\right),\\ u^\star_{\bar{k}k}=\mathbf{u}_{k}\cdot\mathbf{n}_{\bar{k}k}+\mathscr{W}_{k}\left(\mathbf{U}_{k},P^\star_{\bar{k}k}\right),\end{cases}\tag{5.4}$$

where $\mathscr{W}_i(\mathbf{U}_i,P^\star)$ is the wave curve coming from the state $\mathbf{U}_i$ with equation of state i evaluated at a pressure equal to $P^\star$. The intersection of the wave curves https://start.ubuntu-mate.org/ for general equations of state is a nonlinear problem, which can be solved, for example, with the Newton–Raphson method. The *acoustic approximation* consists in linearizing the problem (5.4): instead of intersecting the wave curves, the solution of the Riemann problem is approximated by the intersection of the *linearization of the wave curves*. In this case, the *linearized Riemann problem* becomes

$$\text{Find } \left(u^\star_{\bar{k}k},P^\star_{\bar{k}k}\right)\quad\text{such that}\qquad\begin{cases}u^\star_{\bar{k}k}=\mathbf{u}_{\bar{k}}\cdot\mathbf{n}_{\bar{k}k}-\dfrac{1}{\rho_{\bar{k}}c_{\bar{k}}}\left(P^\star_{\bar{k}k}-P_{\bar{k}}\right)\\ u^\star_{\bar{k}k}=\mathbf{u}_{k}\cdot\mathbf{n}_{\bar{k}k}+\dfrac{1}{\rho_{k}c_{k}}\left(P^\star_{\bar{k}k}-P_{k}\right).\end{cases}$$

This linear system can be solved explicitly, giving the following linear approximation

$$\begin{cases}u^\star_{\bar{k}k}(\mathbf{n}_{\bar{k}k})=\dfrac{Z_k\mathbf{u}_k+Z_{\bar{k}}\mathbf{u}_{\bar{k}}}{Z_k+Z_{\bar{k}}}\cdot\mathbf{n}_{\bar{k}k}+\dfrac{P_{\bar{k}}-P_k}{Z_k+Z_{\bar{k}}}\\ P^\star_{\bar{k}k}(\mathbf{n}_{\bar{k}k})=\dfrac{Z_{\bar{k}}P_k+Z_kP_{\bar{k}}}{Z_k+Z_{\bar{k}}}+\dfrac{Z_kZ_{\bar{k}}(\mathbf{u}_{\bar{k}}-\mathbf{u}_k)}{Z_k+Z_{\bar{k}}}\cdot\mathbf{n}_{\bar{k}k}.\end{cases}$$

Using such an approximation, the integrals involved in Proposition 5.4 can be explicitly computed up to some constants that are not totally explicit.

PROPOSITION 5.5 (Linearized case). *Suppose that the hypotheses of Proposition 5.4 hold. If we suppose that for all* $\mathbf{x}_d$

$$\begin{cases}u^\star_{\bar{k}k}(\mathbf{x}_d)=\dfrac{Z_k\mathbf{u}_k+Z_{\bar{k}}\mathbf{u}_{\bar{k}}}{Z_k+Z_{\bar{k}}}\cdot\dfrac{\mathbf{x}_d}{\|\mathbf{x}_d\|}+\dfrac{P_{\bar{k}}-P_k}{Z_k+Z_{\bar{k}}}\\ P^\star_{\bar{k}k}(\mathbf{x}_d)=\dfrac{Z_{\bar{k}}P_k+Z_kP_{\bar{k}}}{Z_k+Z_{\bar{k}}}+\dfrac{Z_kZ_{\bar{k}}(\mathbf{u}_{\bar{k}}-\mathbf{u}_k)}{Z_k+Z_{\bar{k}}}\cdot\dfrac{\mathbf{x}_d}{\|\mathbf{x}_d\|},\end{cases}$$

where $Z_k = \rho_k c_k$ is the acoustic impedance, then, defining the interfacial linearized expressions

$$\mathbf{u}_I^L = \frac{Z_k \mathbf{u}_k + Z_{\bar{k}} \mathbf{u}_{\bar{k}}}{Z_k + Z_{\bar{k}}}, \qquad P_I^L = \frac{Z_{\bar{k}} P_k + Z_k P_{\bar{k}}}{Z_k + Z_{\bar{k}}},$$

the relaxation coefficient

$$\mathscr{L}(\Lambda) = \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbf{x}_d \in \mathbb{S}^{d-1}} \|\Lambda \mathbf{x}_d\| \, \mathrm{d}\mathbf{x}_d,$$

and the relaxation matrix

$$\tilde{\Lambda} = Q^T \Lambda^2 \left(\frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbf{x}_d \in \mathbb{S}^{d-1}} \frac{\mathbf{y}_d \mathbf{y}_d^T}{\|\Lambda \mathbf{y}_d\|} \mathrm{d}\mathbf{y}_d \right) Q,$$

then system (1.2) is found with the following closure of the interfacial terms,

$$\begin{cases} \mathbf{P}_I = P_I^L \mathrm{I}_d, \\ \mathbf{u}_I = \mathbf{u}_I^L, \\ (P\mathbf{u})_I = \dfrac{(Z_{\bar{k}} P_k + Z_k P_{\bar{k}}) (Z_k \mathbf{u}_k + Z_{\bar{k}} \mathbf{u}_{\bar{k}}) + Z_k Z_{\bar{k}} (P_{\bar{k}} - P_k)(\mathbf{u}_{\bar{k}} - \mathbf{u}_k)}{(Z_k + Z_{\bar{k}})^2}, \end{cases}$$

and the following linear, anisotropic relaxation terms,

$$\begin{cases} \mathcal{R}_k^{(\alpha)} = \dfrac{\exp(-m_k(\mathbf{x})^2/2) \left|\mathbb{S}^{d-1}\right| \mathscr{L}(\Lambda)}{2\pi \left|\mathbb{V}_{d-1}\right|} \dfrac{P_k - P_{\bar{k}}}{Z_k + Z_{\bar{k}}}, \\ \mathcal{R}_k^{(\rho \mathbf{u})} = \dfrac{\exp(-m_k(\mathbf{x})^2/2) \left|\mathbb{S}^{d-1}\right|}{2\pi \left|\mathbb{V}_{d-1}\right|} \dfrac{Z_k Z_{\bar{k}}}{Z_k + Z_{\bar{k}}} \tilde{\Lambda} \left(\mathbf{u}_{\bar{k}} - \mathbf{u}_k\right), \\ \mathcal{R}_k^{(\rho E)} = \mathbf{u}_I^L \cdot \mathcal{R}_k^{(\rho \mathbf{u})} - P_I^L \mathcal{R}_k^{(\alpha)}. \end{cases}$$

The proof consists in computing explicitly the integrals found in Proposition 5.4. The computations are a bit long, and the proof is deferred to Appendix A. As a first order approximation was used for $P^\star$ and $u^\star$, it would then be consistent to neglect the second order term in $(P_{\bar{k}} - P_k)(\mathbf{u}_{\bar{k}} - \mathbf{u}_k)$ of $(P\mathbf{u})_I$, which leads to the interfacial terms of [43, 45]. We remark that with the acoustic approximation, the interfacial terms do not depend on the anisotropy of the distribution, whereas the relaxation parameters depend on this anisotropy. Moreover, referring to subsubsection 3.2.4, we remark that the intensity of the relaxation parameters is linked with the invert of the size of the bubbles, for which a rather good approximation for intermediate α is given by (3.6), and that the anisotropy of the relaxation parameters is directly linked with the anisotropy of the average bubbles.

5.6. Short memory, isotropic, linearized case. When the auto-correlation function is isotropic, the relaxation terms can be further simplified for giving the following.

PROPOSITION 5.6 (Linearized case with isotropic distribution). *In the same conditions as in Proposition 5.4, and if we suppose moreover that the auto-correlation function is isotropic, $\breve{\Sigma} = \nu^2 \mathrm{I}_d$, then the same closure for the interfacial terms as Proposition 5.4 is found, and the relaxation terms can be further simplified for giving a form similar to (1.1) with*

$$\begin{cases} \mu = \dfrac{\exp(-m_k(\mathbf{x})^2/2) \left|\mathbb{S}^{d-1}\right| \nu}{2\pi \left|\mathbb{V}_{d-1}\right|} \dfrac{1}{Z_k + Z_{\bar{k}}}, \\ \lambda = \dfrac{\exp(-m_k(\mathbf{x})^2/2) \left|\mathbb{S}^{d-1}\right| \nu}{2\pi \left|\mathbb{V}_{d-1}\right|} \dfrac{Z_k Z_{\bar{k}}}{Z_k + Z_{\bar{k}}}. \end{cases}$$

Note that in the previous proposition, the relaxation parameters are depending on two types of parameters,

$$\frac{\exp(-m_k(\mathbf{x})^2/2)\left|\mathbb{S}^{d-1}\right|\nu}{2\pi\left|\mathbb{V}_{d-1}\right|},$$

which is characteristic of the mixture of the flow (volume fraction, bubble size), and $\frac{Z_k Z_{\bar{k}}}{Z_k+Z_{\bar{k}}}$ and $\frac{1}{Z_k+Z_{\bar{k}}}$, which are depending on the local state of the two phases. Note that the ratio of the relaxation parameters is consistent with the ones of [44].

6. Properties of the model found.

6.1. Hyperbolicity. The most general system found is (1.2), which was found in the short memory case with a nonlinear flow in Proposition 5.4. It includes also all the models derived in this paper. The hyperbolic part of this system can be written in the following quasilinear form:

$$\partial_t \mathbf{V} + \sum_{i=1}^{d} A_i(\mathbf{V})\partial_{\mathbf{x}_i}\mathbf{V} = 0.$$

In this whole section, we denote by $\boldsymbol{\xi}$ a nontrivial direction in $\mathbb{R}^d$. The Jacobian matrix in the $\boldsymbol{\xi}$ direction $\mathbf{J}_{\boldsymbol{\xi}}$ reads

$$\sum_{i=1}^{d} A_i(\mathbf{V})\boldsymbol{\xi}_i = \left(\begin{array}{c|c|c} \mathbf{u}_I\cdot\boldsymbol{\xi} & 0 & 0 \\ \hline 0 & & \\ \left(P_1+\frac{\rho_1\kappa_1}{\beta_1}\right)\boldsymbol{\xi}-\mathbf{P}_I\boldsymbol{\xi} & \mathrm{Euler}_1(\boldsymbol{\xi}) & 0 \\ \left(P_1+\frac{\rho_1\kappa_1}{\beta_1}\right)\mathbf{u}_1\cdot\boldsymbol{\xi}-(P\mathbf{u})_I\cdot\boldsymbol{\xi} & & \\ \hline 0 & & \\ -\left(P_2+\frac{\rho_2\kappa_2}{\beta_2}\right)\boldsymbol{\xi}+\mathbf{P}_I\boldsymbol{\xi} & 0 & \mathrm{Euler}_2(\boldsymbol{\xi}) \\ -\left(P_2+\frac{\rho_2\kappa_2}{\beta_2}\right)\mathbf{u}_2\cdot\boldsymbol{\xi}+(P\mathbf{u})_I\cdot\boldsymbol{\xi} & & \end{array}\right), \tag{6.1}$$

where $\mathrm{Euler}_k(\boldsymbol{\xi})$ is the Jacobian of the classical Euler equations, equal to

$$\left(\begin{array}{c|c|c} 0 & \boldsymbol{\xi}^T & 0 \\ \hline \left(c_k^2+\frac{|\mathbf{u}_k|^2-H_k}{\rho_k\beta_k}\right)\boldsymbol{\xi}-(\mathbf{u}_k\cdot\boldsymbol{\xi})\mathbf{u}_k & \mathbf{u}_k^T\boldsymbol{\xi}-\frac{\boldsymbol{\xi}^T\mathbf{u}_k}{\rho_k\beta_k}+(\mathbf{u}_k\cdot\boldsymbol{\xi})\mathrm{I}_d & \frac{\boldsymbol{\xi}}{\rho_k\beta_k} \\ \hline \left(c_k^2+\frac{|\mathbf{u}_k|^2-H_k}{\rho_k\beta_k}-H_k\right)\mathbf{u}_k\cdot\boldsymbol{\xi} & H_k\boldsymbol{\xi}^T-\frac{\mathbf{u}_k\cdot\mathbf{n}}{\rho_k\beta_k}\mathbf{u}_k^T & \left(1+\frac{1}{\rho_k\beta_k}\right)\mathbf{u}_k\cdot\boldsymbol{\xi} \end{array}\right),$$

where the following notation was used

$$\beta_k = \left(\frac{\partial\varepsilon_k}{\partial P_k}\right)_{\rho_k}, \qquad \kappa_k = \left(\frac{\partial\varepsilon_k}{\partial\rho_k}\right)_{P_k}, \qquad H_k = E_k + \frac{P_k}{\rho_k}.$$

Clearly, the Jacobian of the system has real eigenvalues, which are the eigenvalues of the Euler parts $\mathbf{u}_i\cdot\boldsymbol{\xi}$ with multiplicity d and $\mathbf{u}_i\cdot\boldsymbol{\xi}\pm c_i\|\boldsymbol{\xi}\|$ with multiplicity 1, and $\mathbf{u}_I\cdot\boldsymbol{\xi}$ with multiplicity one. The minimum number of eigenvectors of (1.2) is given by the following.

PROPOSITION 6.1 (Minimum number of eigenvectors). *In the direction $\boldsymbol{\xi}$, the system has at least $2(d+2)$ free right eigenvectors, which are associated with the eigenvalues $\mathbf{u}_k \cdot \boldsymbol{\xi}$ with multiplicity d, and $\mathbf{u}_k \cdot \boldsymbol{\xi} \pm c_k \|\boldsymbol{\xi}\|$ with multiplicity 1.*

Proof. Considering the block structure of the Jacobian matrix, the vectors

- $(0, \check{r}_1(\boldsymbol{\xi}), 0)$, where $\check{r}_1(\boldsymbol{\xi})$ is one of the $(d+2)$ right eigenvector of $\text{Euler}_1(\boldsymbol{\xi})$,
- $(0, 0, \check{r}_2(\boldsymbol{\xi}))$, where $\check{r}_2(\boldsymbol{\xi})$ is one of the $(d+2)$ right eigenvector of $\text{Euler}_2(\boldsymbol{\xi})$

are a system of $2(d+2)$ free eigenvectors of the Jacobian matrix, which ends the proof. ∎

As a consequence, when $\mathbf{u}_I \cdot \boldsymbol{\xi} \neq \mathbf{u}_k \cdot \boldsymbol{\xi}$ and $\mathbf{u}_I \cdot \boldsymbol{\xi} \neq \mathbf{u}_k \cdot \boldsymbol{\xi} \pm c_k \|\boldsymbol{\xi}\|$, the system has a full set of eigenvectors and is hyperbolic. However, the system is only weakly hyperbolic in general, as given by the following proposition.

PROPOSITION 6.2 (Hyperbolicity of the system).

- *If for $i = 1, 2$, $\mathbf{u}_I \cdot \boldsymbol{\xi} \neq \mathbf{u}_i \cdot \boldsymbol{\xi}$ and $\mathbf{u}_I \cdot \boldsymbol{\xi} \neq \mathbf{u}_i \cdot \boldsymbol{\xi} \pm c_i \|\boldsymbol{\xi}\|$, then the system is hyperbolic.*
- *If $\mathbf{u}_I \cdot \boldsymbol{\xi} = \mathbf{u}_k \cdot \boldsymbol{\xi}$ and $\mathbf{u}_I \cdot \boldsymbol{\xi} \neq \mathbf{u}_{\bar{k}} \cdot \boldsymbol{\xi}$, $\mathbf{u}_I \cdot \boldsymbol{\xi} \neq \mathbf{u}_{\bar{k}} \cdot \boldsymbol{\xi} \pm c_{\bar{k}} \|\boldsymbol{\xi}\|$ then the system is hyperbolic if and only if*
  1. *for all ζ normal to $\boldsymbol{\xi}$, $\boldsymbol{\zeta}^T \mathbf{P}_I \boldsymbol{\xi} = 0$.*
  2. *$\mathbf{u}_k^T \mathbf{P}_I \boldsymbol{\xi} - (P\mathbf{u})_I \cdot \boldsymbol{\xi} = 0$.*
- *If $\mathbf{u}_I \cdot \boldsymbol{\xi} = \mathbf{u}_k \cdot \boldsymbol{\xi} + c_k \|\boldsymbol{\xi}\|$, and $\mathbf{u}_I \cdot \boldsymbol{\xi} \neq \mathbf{u}_{\bar{k}} \cdot \boldsymbol{\xi}$, $\mathbf{u}_I \cdot \boldsymbol{\xi} \neq \mathbf{u}_{\bar{k}} \cdot \boldsymbol{\xi} \pm c_{\bar{k}} \|\boldsymbol{\xi}\|$, then the system is hyperbolic if and only if*

$$-\left(\rho_1 c_1 \left(P_1 \beta_1 + \kappa_1 \rho_1\right) \|\boldsymbol{\xi}\|^2 - \beta_1 \boldsymbol{\xi}^T \mathbf{P}_I \boldsymbol{\xi}\right) + \|\boldsymbol{\xi}\| \left((P\mathbf{u})_I \cdot \boldsymbol{\xi} - \mathbf{u}_1^T \mathbf{P}_I \boldsymbol{\xi}\right) = 0. \tag{6.2}$$

The proof is a bit long and is deferred to Appendix B. We remark that if the two conditions of hyperbolicity when $\mathbf{u}_I \cdot \boldsymbol{\xi} = \mathbf{u}_k \cdot \boldsymbol{\xi}$ are ensured in all the directions $\boldsymbol{\xi}$, then the first one implies that $\mathbf{P}_I$ is isotropic, whereas the second one implies that $(P\mathbf{u})_I = P_I \mathbf{u}_k = P_I \mathbf{u}_I$. Isotropy of the interfacial pressure is ensured for the model of Proposition 5.6, but the second property is not ensured. This means that depending on the closure, the case $\mathbf{u}_I \cdot \boldsymbol{\xi} = \mathbf{u}_1 \cdot \boldsymbol{\xi}$ may be resonant if $\mathbf{P}_I$ is not isotropic, or may not be resonant otherwise.
When $\mathbf{u}_I \cdot \boldsymbol{\xi}$ crosses $\mathbf{u}_1 \cdot \boldsymbol{\xi} + c_1 \|\boldsymbol{\xi}\|$, and if we additionally suppose that $\mathbf{P}_I$ is isotropic, that $(P\mathbf{u})_I = P_I \mathbf{u}_I$, and that the fluid 1 follows a perfect gas equation of state, then (6.2) is proportional to $\beta_1 + c_1$ which never vanishes. We conclude that in general, the case $\mathbf{u}_I \cdot \boldsymbol{\xi} = \mathbf{u}_1 \cdot \boldsymbol{\xi} + c_1 \|\boldsymbol{\xi}\|$ is resonant.

6.2. Nature of the field $\mathbf{u}_I \cdot \boldsymbol{\xi}$. Most of the numerical schemes for hyperbolic systems rely on the finite volume method, in which the Riemann problem can be seen as a building block. For solving it, an important property to check for each field is whether it is *genuinely nonlinear* or *linearly degenerate* (see [22, p.41]). In the first case, it may induce a shock or a rarefaction wave, whereas in the latter, the solution of the Riemann problem through this wave is a discontinuity moving at the velocity of the field. This concept is even more important for systems like (1.1),(1.2) which cannot be written in a conservative form: if the field is linearly degenerate, then Rankine–Hugoniot relations across discontinuities can be defined, whereas if the field is genuinely nonlinear, it is not possible. This raises also problems from a numerical point of view, for example when nonconservative methods are used for conservative systems, see [1, 28]. In the case of essentially nonconservative systems, usual numerical schemes applied to a genuinely nonlinear field involving nonconservative products usually lead in fact in a *nonconvergent* scheme. This fact is unfortunately rarely reported, but for multiphase flows, this was very well illustrated in [23]: for a closure

for which the fluid $\mathbf{u}\cdot\boldsymbol{\xi}$ is genuinely nonlinear [21], the VFRoe numerical scheme is not convergent [23, Fig. 8.3 p. 135], whereas it is convergent for [10], a closure where $(\mathbf{u}_I\cdot\boldsymbol{\xi})$ is linearly degenerate [23, Fig. 8.7 p. 143].
As far as the closures that were introduced in this article are concerned, the one of Proposition 5.2 was discussed in [17] and gives a genuinely nonlinear field. For the closure of Proposition 5.5, tedious calculation performed with perfect gas equation of state proves that this field is also genuinely nonlinear in general. We did not investigate if the additional degree of freedom given by $(P\mathbf{u})_I \neq P_I\mathbf{u}_I$ could be used for enforcing the linearly degeneracy of the field $\mathbf{u}_I$.

6.3. Phase entropy dissipation. If s_k denotes the specific entropy of the phase k, then $(-\rho_k s_k)$ is a mathematical entropy for the Euler system (2.2), which we denote by S_k. One characteristic of the system (1.1) is that it does not have an entropy in general. In this section, we are not interested in the existence of a mathematical entropy for the system (1.1) or (1.2), but rather in the dissipation property of the phase entropy $\alpha_k S_k$. A first step is to find the phase entropy equation.

PROPOSITION 6.3 (Phase entropy equation). *For (1.2), the following phase entropy equation holds:*

$$\begin{aligned}\partial_t\left(\alpha_k S_k(\mathbf{U}_k)\right)+\operatorname{div}_{\mathbf{x}}\left(\alpha_k S_k(\mathbf{U}_k)\right)\\ +\frac{1}{T_k}\left((P\mathbf{u})_I+P_k\mathbf{u}_k-P_k\mathbf{u}_I-\mathbf{u}_k\mathbf{P}_I\right)\nabla\alpha_k\\ =\frac{1}{T_k}\left(-P_k\mathcal{R}_k^{(\alpha)}+\mathbf{u}_k\cdot\mathcal{R}_k^{(\rho\mathbf{u})}-\mathcal{R}_k^{(\rho E)}\right).\end{aligned}\tag{6.3}$$

Proof. $S=-\rho s$ is a mathematical entropy for the Euler system. This defines the following entropic variables:

$$S_{\mathbf{U}}=\frac{1}{T}\begin{pmatrix} g-\frac{1}{2}|\mathbf{u}|^2\\ \mathbf{u}\\ -1\end{pmatrix},$$

where g is the free enthalpy

$$g:=\varepsilon+\frac{P}{\rho}-Ts.$$

For regular flows, the conservative equation on the entropy is obtained by multiplying on the left the conservative system by $S_{\mathbf{U}}$. This gives

$$\partial_t S(\mathbf{U})+\operatorname{div}_{\mathbf{x}}\left(S(\mathbf{U})\mathbf{u}\right)=0.$$

Remarking that

$$\left(S_{\mathbf{U}}\right)^T\mathbf{U}=\frac{P}{T}+S(\mathbf{U})\qquad\text{and}\qquad\left(S_{\mathbf{U}}\right)^T\mathbf{F}(\mathbf{U})=\left(\frac{P}{T}+S(\mathbf{U})\right)\mathbf{u},$$

we get, for any α_k

$$\begin{aligned}
\left(S_{\mathbf{U}}\right)^T \left(\partial_t \left(\alpha_k \mathbf{U}\right) + \operatorname{div}_{\mathbf{x}} \left(\alpha_k \mathbf{F}(\mathbf{U})\right)\right) &= \alpha_k \left(S_{\mathbf{U}}\right)^T \left(\partial_t \mathbf{U} + \operatorname{div}_{\mathbf{x}} \left(\mathbf{F}(\mathbf{U})\right)\right) \\
&\quad + \left(S_{\mathbf{U}}\right)^T \mathbf{U} \partial_t \alpha_k + \left(S_{\mathbf{U}}\right)^T \mathbf{F}(\mathbf{U}) \nabla \alpha_k \\
&= \alpha_k \left(\partial_t (S(\mathbf{U})) + \operatorname{div}_{\mathbf{x}} (S(\mathbf{U})\mathbf{u})\right) \\
&\quad + \left(\frac{P}{T} + S(\mathbf{U})\right) \left(\partial_t \alpha_k + \mathbf{u} \cdot \nabla \alpha_k\right) \\
&= \partial_t (\alpha_k S(\mathbf{U})) + \operatorname{div}_{\mathbf{x}} (\alpha_k S(\mathbf{U})\mathbf{u}) \\
&\quad + \frac{P}{T} \left(\partial_t \alpha_k + \mathbf{u} \cdot \nabla \alpha_k\right).
\end{aligned}$$

Now, using this last equation with model (1.2) gives for the fluid k

$$\begin{aligned}
&\partial_t \left(\alpha_k S_k(\mathbf{U}_k)\right) + \operatorname{div}_{\mathbf{x}} \left(\alpha_k S_k(\mathbf{U}_k)\right) \\
&\quad + \frac{1}{T_k} \left((P\mathbf{u})_I + P_k \mathbf{u}_k - P_k \mathbf{u}_I - \mathbf{u}_k \mathbf{P}_I\right) \nabla \alpha_k \\
&\qquad = \frac{1}{T_k} \left(-P_k \mathcal{R}_k^{(\alpha)} + \mathbf{u}_k \cdot \mathcal{R}_k^{(\rho \mathbf{u})} - \mathcal{R}_k^{(\rho E)}\right).
\end{aligned}$$

□

The phase entropy equation (6.3) includes three types of contributions: the phase entropy conservation contribution (first line of (6.3)), the nonconservative contribution (second line of (6.3)), and the relaxation contribution (third line of (6.3)).

It is usual to expect entropy dissipation coming from the relaxation terms, however, entropy dissipation may come from the nonconservative part also, as was remarked on [12]. This is actually something that is common to some of the models derived in section 5, as summarized in the following proposition

PROPOSITION 6.4 (Entropy dissipation of the nonconservative part). *For the nonconservative part defined in Proposition 5.2, the nonconservative part of the entropy equation is dissipative:*

$$\frac{1}{T_k} \left((P\mathbf{u})_I + P_k \mathbf{u}_k - P_k \mathbf{u}_I - \mathbf{u}_k \mathbf{P}_I\right) \cdot \nabla \alpha_k \geq 0.$$

For (1.2), if $\mathbf{u}_I = Y_k \mathbf{u}_k + Y_{\bar{k}} \mathbf{u}_{\bar{k}}$ with $Y_k + Y_{\bar{k}} = 1$ and if $\mathbf{P}_I$ is isotropic, then the nonconservative part of the entropy dissipation is 0 if and only if

$$\begin{cases} P_I = Y_{\bar{k}} P_k + Y_k P_{\bar{k}}, \\ (P\mathbf{u})_I = Y_k P_{\bar{k}} \mathbf{u}_k + Y_{\bar{k}} P_k \mathbf{u}_{\bar{k}}. \end{cases} \tag{6.4}$$

The full proof was deferred in subsection C.1. Note that for the closure proposed in Proposition 5.4, the interfacial quantities do not depend on the gradient of the volume fraction. As a consequence, the nonconservative part of the entropy equation can be not dissipative. For the same reason, the nonconservative part of Proposition 5.3, cannot be entropy dissipative, because it gives Proposition 5.4 in some limits.

In general, the closures proposed in Proposition 5.5 and Proposition 5.6 do not ensure the condition (6.4). We remark that a closure giving a nonconservative contribution for the entropy equal to 0 is the objective of [12] and see here that this objective can be fulfilled more easily provided the constraint on $(P\mathbf{u})_I$ is relaxed (i.e. $(P\mathbf{u})_I \neq P_I \mathbf{u}_I$).

Last, we prove that the relaxation terms also contribute to the entropy dissipation.

Models	Hyperbolic terms	Relaxation terms
Long memory model	✓	✓
One dimensional model	No	✓
Short memory model	No	✓
Short memory, linearized model	No	✓

TABLE 6.1
Summary of the phase entropy dissipation results for the different systems found.

PROPOSITION 6.5 (Entropy dissipation of the relaxation terms). *For the relaxation terms defined in Proposition 5.3, Proposition 5.4, Proposition 5.5,*

$$\frac{1}{T_k}\left(-P_k\mathcal{R}_k^{(\alpha)} + \mathbf{u}_k\cdot\mathcal{R}_k^{(\rho\mathbf{u})} - \mathcal{R}_k^{(\rho E)}\right) \leq 0.$$

The proof of this proposition is also given as an annex in subsection C.2. The results concerning the entropy dissipation were summarized in Table 6.1. We remark that the nonlinear model proposed in Proposition 5.4 does not ensure the entropy inequality for the nonconservative terms, but only for the relaxation terms. However, we recall that Proposition 5.4 relies on an asymptotic development of (4.3) when $\check{\Sigma}$ is greater than the gradient of the volume fraction. In this asymptotic regime, the zeroth order terms are the relaxation terms, whereas the nonconservative products are the first order terms, (see (5.2)). This implies that in this regime, the relaxation terms are one order of magnitude larger than the nonconservative products.

Last, we remark that only the phase entropy dissipation was addressed in this section. The equation on the full entropy can be obtained by adding the two phase entropy equations. For the long memory case, the full entropy is dissipative. In the other cases, the nonconservative part does not vanish and has no sign, and the relaxation terms are entropy dissipative. Still, the remark on the relative magnitude of the relaxation terms and nonconservative terms states that the dissipativity of the relaxation terms should be enough for ensuring the full entropy dissipation.

6.4. Symmetrizability of (1.2). A weaker form of the entropy existence is the symmetrizability of the system. Conditions for symmetrizability of the system (1.2) are summarized in the following proposition

PROPOSITION 6.6. *The system (1.2) is not symmetrizable in conservative variables in general.* The proof of Proposition 6.6 is detailed in Appendix D.

7. Conclusion. In this article, a general framework was proposed for deriving two-phase models by homogenization of Euler equations. This framework relies on two ingredients:

- the knowledge of the local exact solution, based on the solution of the Riemann problem,
- the definition of an explicit stochastic model for the spatial phase distribution. Choosing a model based on a Gaussian process allows to perform explicit computations based on few parameters, which, in our case is only the second order derivative of the auto-correlation function. The physical relevance of this model was discussed in section 3.

Thanks to this method, we managed to

- find again the closure previously proposed in [17, 2, 8], but by giving a clear topological context of this closure in Proposition 5.2,

- propose a calibration of the relaxation terms for (1.1) based on the local average topology of the flow in Proposition 5.6, which may include the anisotropy of the bubbles and droplets in Proposition 5.5,
- propose a new model (1.2), which is a generalization of (1.1) when the flow is still well mixed, but nonlinear.

Beyond the new models derived, we would like to emphasize that the method clarifies the hypothesis on the local topology of the flows which lead to these averaged models. We would like also to point out that a general outcome of our method is that relaxation terms and interfacial terms both come from the averaging of the same term (4.3), so that they cannot be considered separately: for example, the closure found in Proposition 5.2 matches with well separated interfaces, and so the relaxation terms are 0, whereas Proposition 5.6 matches with well mixed flows, for which relaxation parameters are large.

We see at least two directions for further developing this model:

- The derivation of the averaged system by homogenization was based on the Euler model without surface tension between the phases. We plan to perform similar computations including surface tension effects inside the local Riemann problem. Also, diffusive terms, namely, using Navier-Stokes equations instead of Euler equations, can be considered for including viscous effects in the closure and relaxation terms. In this article, the relaxation terms are induced by the nonlinear effects of the Euler system, which tend to induce equality of pressures and normal velocities at a contact and depend on acoustic coefficients in the two fluids. If additionaly Navier-Stokes terms are taken into account, they will tend to induce equality of temperatures and full velocities at a contact through relaxation terms depending on the heat conduction and viscosity. When considering viscous effects, relaxation terms similar to the ones proposed in [6] should be found, at least in some regimes.
- The derivation was led with only two phases. By developing another stochastic model which would include more than two phases, we should be able to derive a model with an arbitrary number of phases. For the moment, we have no idea on how to model a flow with more than two phases with a stochastic model.

Further extensions include the derivation of numerical schemes for the new systems developed. A natural framework for developing numerical schemes based on a stochastic model is [2]. This method depends on the definition of weights which are characteristic of the topology of the flow. More precisely, the weights proposed in [2] were proved to match with the long memory case (i.e. interface flows) in [17]. By using a stochastic model with a different covariance function, which will lead to other weights, it is possible to extend the numerical method developed in [2, 17] to different regimes of multiphase flows.

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Appendix A. Details of the computations for Proposition 5.5.

For the sake of simplicity, we rewrite the pressure or the velocity linearization in the acoustic regime as

$$\frac{\boldsymbol{\alpha}^T \mathbf{x}_d}{\|\mathbf{x}_d\|} + \beta.$$

A.1. Interfacial velocity computation. We want to compute

$$\int_{\mathbf{x}_d \in Q^T \Lambda(\mathbb{S}^{d-1})} u^\star_{\bar{k}k}\left(\frac{\mathbf{x}_d}{\|\mathbf{x}_d\|}\right) \|\mathbf{x}_d\| \, \mathbf{x}_d \mathrm{d}\mathbf{x}_d$$

with $u^\star_{\bar{k}k}\left(\dfrac{\mathbf{x}_d}{\|\mathbf{x}_d\|}\right) = \dfrac{\boldsymbol{\alpha}^T \mathbf{x}_d}{\|\mathbf{x}_d\|} + \beta$. We then have to evaluate

$$\int_{\mathbf{x}_d \in Q^T \Lambda(\mathbb{S}^{d-1})} \left(\boldsymbol{\alpha}^T \mathbf{x}_d\right) \mathbf{x}_d \mathrm{d}\mathbf{x}_d + \int_{\mathbf{x}_d \in Q^T \Lambda(\mathbb{S}^{d-1})} \beta \|\mathbf{x}_d\| \, \mathbf{x}_d \mathrm{d}\mathbf{x}_d,$$

which can be rewritten

$$\left(\int_{\mathbf{x}_d \in Q^T \Lambda(\mathbb{S}^{d-1})} \mathbf{x}_d \mathbf{x}_d^T \mathrm{d}\mathbf{x}_d\right) \boldsymbol{\alpha} + \beta \left(\int_{\mathbf{x}_d \in Q^T \Lambda(\mathbb{S}^{d-1})} \|\mathbf{x}_d\| \, \mathbf{x}_d \mathrm{d}\mathbf{x}_d\right).$$

By symmetry, the integral after β is 0. For the integral in front of $\boldsymbol{\alpha}$, which we denote by $\mathscr{I}_1$, we have

$$\begin{aligned}
\mathscr{I}_1 &= \int_{\mathbf{x}_d \in Q^T \Lambda(\mathbb{S}^{d-1})} \mathbf{x}_d \mathbf{x}_d^T \mathrm{d}\mathbf{x}_d \\
&= \int_{\mathbf{y}_d \in \mathbb{S}^{d-1}} \left(Q^T \Lambda \mathbf{y}_d\right) \left(Q^T \Lambda \mathbf{y}_d\right)^T \left|\det \check{\Sigma}\right|^{1/2} \mathrm{d}\mathbf{y}_d \\
\mathscr{I}_1 &= Q^T \Lambda \left(\int_{\mathbf{y}_d \in \mathbb{S}^{d-1}} \mathbf{y}_d \mathbf{y}_d^T \mathrm{d}\mathbf{y}_d\right) \Lambda Q \left|\det \check{\Sigma}\right|^{1/2},
\end{aligned}$$

and it can be proved that

$$\int_{\mathbf{y}_d \in \mathbb{S}^{d-1}} \mathbf{y}_d \mathbf{y}_d^T \mathrm{d}\mathbf{y}_d = |\mathbb{V}_d| \, \mathrm{I}_d.$$

We finally find that

$$\mathscr{I}_1 = |\mathbb{V}_d| \left|\det \check{\Sigma}\right|^{1/2} \check{\Sigma}$$

and that

$$\mathbf{u}_I = \frac{Z_k \mathbf{u}_k + Z_{\bar{k}} \mathbf{u}_{\bar{k}}}{Z_k + Z_{\bar{k}}}.$$

A.2. Interfacial pressure tensor . We want to compute

$$\int_{\mathbf{x}_d \in Q^T \Lambda(\mathbb{S}^{d-1})} P^\star_{\bar{k}k}\left(\frac{\mathbf{x}_d}{\|\mathbf{x}_d\|}\right) \mathbf{x}_d \mathbf{x}_d^T \, \mathrm{d}\mathbf{x}_d$$

with $P^\star_{\bar{k}k}\left(\dfrac{\mathbf{x}_d}{\|\mathbf{x}_d\|}\right) = \dfrac{\boldsymbol{\alpha}^T \mathbf{x}_d}{\|\mathbf{x}_d\|} + \beta$. We then have to evaluate

$$\int_{\mathbf{x}_d \in Q^T \Lambda(\mathbb{S}^{d-1})} \frac{\left(\boldsymbol{\alpha}^T \mathbf{x}_d\right)}{\|\mathbf{x}_d\|} \mathbf{x}_d \mathbf{x}_d^T \mathrm{d}\mathbf{x}_d + \beta \int_{\mathbf{x}_d \in Q^T \Lambda(\mathbb{S}^{d-1})} \mathbf{x}_d \mathbf{x}_d^T \mathrm{d}\mathbf{x}_d.$$

The integral involving $\boldsymbol{\alpha}$ is zero, by symmetry. We are then interested in the integral in front of β, which we denote by $\mathscr{I}_2$. Then

$$\begin{aligned}
\mathscr{I}_2 &= \int_{\mathbf{x}_d \in Q^T\Lambda(\mathbb{S}^{d-1})} \mathbf{x}_d \mathbf{x}_d^T \mathrm{d}\mathbf{x}_d \\
&= \int_{\mathbf{y}_d \in \mathbb{S}^{d-1}} Q^T \Lambda \mathbf{y}_d \left(Q^T \Lambda \mathbf{y}_d\right)^T \left|\det \check{\Sigma}\right|^{1/2} \mathrm{d}\mathbf{y}_d \\
&= \left|\det \check{\Sigma}\right|^{1/2} Q^T \Lambda \left(\int_{\mathbf{y}_d \in \mathbb{S}^{d-1}} \mathbf{y}_d \mathbf{y}_d^T \mathrm{d}\mathbf{y}_d \right) \Lambda Q \\
\mathscr{I}_2 &= \left|\det \check{\Sigma}\right|^{1/2} |\mathbb{V}_d| \, \check{\Sigma}.
\end{aligned}$$

We finally find that

$$\mathbf{P}_I = \frac{Z_{\bar{k}} P_k + Z_k P_{\bar{k}}}{Z_k + Z_{\bar{k}}} \mathrm{I}_d.$$

A.3. Interfacial $(P\mathbf{u})$. We are looking for the value of the integral

$$\int_{\mathbf{x}_d \in Q^T\Lambda(\mathbb{S}^{d-1})} (Pu)^\star_{\bar{k}k} \left(\frac{\mathbf{x}_d}{\|\mathbf{x}_d\|} \right) \|\mathbf{x}_d\| \, \mathbf{x}_d \mathrm{d}\mathbf{x}_d.$$

Supposing that both $P^\star_{\bar{k}k}$ and $u^\star_{\bar{k}k}$ are linear

$$\begin{aligned}
P^\star_{\bar{k}k} \left(\frac{\mathbf{x}_d}{\|\mathbf{x}_d\|} \right) &= \boldsymbol{\alpha}_0^T \frac{\mathbf{x}_d}{\|\mathbf{x}_d\|} + \beta_0, \\
u^\star_{\bar{k}k} \left(\frac{\mathbf{x}_d}{\|\mathbf{x}_d\|} \right) &= \boldsymbol{\alpha}_1^T \frac{\mathbf{x}_d}{\|\mathbf{x}_d\|} + \beta_1,
\end{aligned}$$

and then

$$(Pu)^\star_{\bar{k}k} \left(\frac{\mathbf{x}_d}{\|\mathbf{x}_d\|} \right) = \boldsymbol{\alpha}_0^T \frac{\mathbf{x}_d \mathbf{x}_d^T}{\|\mathbf{x}_d\|^2} \boldsymbol{\alpha}_1 + (\beta_1 \boldsymbol{\alpha}_0 + \beta_0 \boldsymbol{\alpha}_1)^T \frac{\mathbf{x}_d}{\|\mathbf{x}_d\|} + \beta_0 \beta_1.$$

By symmetry, the integral involving $\dfrac{\mathbf{x}_d \mathbf{x}_d^T}{\|\mathbf{x}_d\|^2}$ is zero. The remaining part is the same as the integral computed for the interfacial velocity. We then find that

$$(P\mathbf{u})_I = \frac{(Z_{\bar{k}} P_k + Z_k P_{\bar{k}}) (Z_k \mathbf{u}_k + Z_{\bar{k}} \mathbf{u}_{\bar{k}}) + Z_k Z_{\bar{k}} (P_{\bar{k}} - P_k)(\mathbf{u}_{\bar{k}} - \mathbf{u}_k)}{(Z_k + Z_{\bar{k}})^2}.$$

A.4. Relaxation in volume fraction. We want to compute

$$\int_{\mathbf{x}_d \in Q^T\Lambda(\mathbb{S}^{d-1})} (-u^\star_{\bar{k}k}) \left(\frac{\mathbf{x}_d}{\|\mathbf{x}_d\|} \right) \|\mathbf{x}_d\| \, \mathrm{d}\mathbf{x}_d$$

with $u^\star_{\bar{k}k} \left(\dfrac{\mathbf{x}_d}{\|\mathbf{x}_d\|} \right) = \dfrac{\boldsymbol{\alpha}^T \mathbf{x}_d}{\|\mathbf{x}_d\|} + \beta$. We then have to evaluate

$$-\int_{\mathbf{x}_d \in Q^T\Lambda(\mathbb{S}^{d-1})} \left(\boldsymbol{\alpha}^T \mathbf{x}_d\right) \mathrm{d}\mathbf{x}_d - \int_{\mathbf{x}_d \in Q^T\Lambda(\mathbb{S}^{d-1})} \beta \|\mathbf{x}_d\| \, \mathrm{d}\mathbf{x}_d.$$

The integral in front of $\boldsymbol{\alpha}$ is zero by symmetry. In the second integral, we make the same variable change as usual for finding

$$\int_{\mathbf{x}_d \in Q^T\Lambda(\mathbb{S}^{d-1})} \|\mathbf{x}_d\| \, \mathrm{d}\mathbf{x}_d = \left(\int_{\mathbf{y}_d \in \mathbb{S}^{d-1}} \|\Lambda \mathbf{x}_d\| \, \mathrm{d}\mathbf{x}_d \right) \left|\det \check{\Sigma}\right|^{1/2}.$$

This integral cannot be explicitly computed in general. Defining

$$\mathscr{L}(\Lambda) = \frac{1}{|\mathbb{S}^{d-1}|}\int_{\mathbf{x}_d\in\mathbb{S}^{d-1}} \|\Lambda\mathbf{x}_d\|\ \mathrm{d}\mathbf{x}_d,$$

this leads to the following relaxation term for the volume fraction:

$$\mathcal{R}_k^{(\alpha)} = \frac{\exp(-m_k(\mathbf{x})^2/2)\left|\mathbb{S}^{d-1}\right|\mathscr{L}(\Lambda)}{2\pi\left|\mathbb{V}_{d-1}\right|}\frac{P_k - P_{\bar{k}}}{Z_k + Z_{\bar{k}}}.$$

A.5. Relaxation in momentum. In this case, we aim at computing

$$\int_{\mathbf{x}_d\in Q^T\Lambda(\mathbb{S}^{d-1})} P^\star_{\bar{k}k}\left(\frac{\mathbf{x}_d}{\|\mathbf{x}_d\|}\right)\mathbf{x}_d\mathrm{d}\mathbf{x}_d$$

with $P^\star_{\bar{k}k}\left(\dfrac{\mathbf{x}_d}{\|\mathbf{x}_d\|}\right) = \dfrac{\boldsymbol{\alpha}^T\mathbf{x}_d}{\|\mathbf{x}_d\|} + \beta$. We then have to evaluate

$$\int_{\mathbf{x}_d\in Q^T\Lambda(\mathbb{S}^{d-1})}\frac{\left(\boldsymbol{\alpha}^T\mathbf{x}_d\right)}{\|\mathbf{x}_d\|}\mathbf{x}_d\mathrm{d}\mathbf{x}_d + \beta\int_{\mathbf{x}_d\in Q^T\Lambda(\mathbb{S}^{d-1})}\mathbf{x}_d\mathrm{d}\mathbf{x}_d.$$

As previously, the integral in front of β is 0, and we denote by $\mathscr{I}_3$ the integral in front of α. Then

$$\begin{aligned}\mathscr{I}_3 &= \left(\int_{\mathbf{x}_d\in Q^T\Lambda(\mathbb{S}^{d-1})}\frac{\mathbf{x}_d\mathbf{x}_d^T}{\|\mathbf{x}_d\|}\mathrm{d}\mathbf{x}_d\right)\boldsymbol{\alpha}\\ &= Q^T\Lambda\left(\int_{\mathbf{x}_d\in\mathbb{S}^{d-1}}\frac{\mathbf{y}_d\mathbf{y}_d^T}{\|\Lambda\mathbf{y}_d\|}\mathrm{d}\mathbf{y}_d\right)\Lambda Q\left|\det\check{\Sigma}\right|^{1/2}.\end{aligned}$$

The matrix $\displaystyle\int_{\mathbf{x}_d\in\mathbb{S}^{d-1}}\frac{\mathbf{y}_d\mathbf{y}_d^T}{\|\Lambda\mathbf{y}_d\|}\mathrm{d}\mathbf{y}_d$ is diagonal. Defining

$$\tilde{\Lambda} := Q^T\Lambda^2\left(\frac{1}{|\mathbb{S}^{d-1}|}\int_{\mathbf{x}_d\in\mathbb{S}^{d-1}}\frac{\mathbf{y}_d\mathbf{y}_d^T}{\|\Lambda\mathbf{y}_d\|}\mathrm{d}\mathbf{y}_d\right)Q,$$

we then find that

$$\mathcal{R}_k^{(\rho\mathbf{u})} = \frac{\exp(-m_k(\mathbf{x})^2/2)\left|\mathbb{S}^{d-1}\right|}{2\pi\left|\mathbb{V}_{d-1}\right|}\frac{Z_kZ_{\bar{k}}}{Z_k+Z_{\bar{k}}}\tilde{\Lambda}\left(\mathbf{u}_{\bar{k}} - \mathbf{u}_k\right).$$

A.6. Relaxation in energy. In this case, we want to compute

$$\int_{\mathbf{x}_d\in Q^T\Lambda(\mathbb{S}^{d-1})}(Pu)^\star_{\bar{k}k}\left(\frac{\mathbf{x}_d}{\|\mathbf{x}_d\|}\right)\|\mathbf{x}_d\|\,\mathrm{d}\mathbf{x}_d$$

with

$$(Pu)^\star_{\bar{k}k}\left(\frac{\mathbf{x}_d}{\|\mathbf{x}_d\|}\right) = \boldsymbol{\alpha}_0^T\frac{\mathbf{x}_d\mathbf{x}_d^T}{\|\mathbf{x}_d\|^2}\boldsymbol{\alpha}_1 + \left(\beta_1\boldsymbol{\alpha}_0 + \beta_0\boldsymbol{\alpha}_1\right)^T\frac{\mathbf{x}_d}{\|\mathbf{x}_d\|} + \beta_0\beta_1.$$

Thus, we want to evaluate

$$\begin{aligned}\boldsymbol{\alpha}_0^T\left(\int_{\mathbf{x}_d\in Q^T\Lambda(\mathbb{S}^{d-1})}\frac{\mathbf{x}_d\mathbf{x}_d^T}{\|\mathbf{x}_d\|}\,\mathrm{d}\mathbf{x}_d\right)\boldsymbol{\alpha}_1 + \left(\beta_1\boldsymbol{\alpha}_0+\beta_0\boldsymbol{\alpha}_1\right)^T\left(\int_{\mathbf{x}_d\in Q^T\Lambda(\mathbb{S}^{d-1})}\mathbf{x}_d\,\mathrm{d}\mathbf{x}_d\right)\\ +\beta_0\beta_1\left(\int_{\mathbf{x}_d\in Q^T\Lambda(\mathbb{S}^{d-1})}\|\mathbf{x}_d\|\ \mathrm{d}\mathbf{x}_d\right).\end{aligned}$$

The first integral is similar to the one computed for the relaxation in momentum, the second one is zero by symmetry, and the third one is similar to the one computed for the relaxation in volume fraction. This finally gives

$$\mathcal{R}_k^{(\rho E)} = \mathbf{u}_I^T \mathcal{R}_k^{(\rho \mathbf{u})} - \frac{Z_{\bar{k}} P_k + Z_k P_{\bar{k}}}{Z_k + Z_{\bar{k}}} \mathcal{R}_k^{(\alpha)}.$$

Appendix B. Details of the proof of Proposition 6.2 . In general, defining

$$K_i = \rho_i \beta_i (\mathbf{u}_I \cdot \boldsymbol{\xi} - \mathbf{u}_i \cdot \boldsymbol{\xi} + c_i \|\boldsymbol{\xi}\|)(\mathbf{u}_I \cdot \boldsymbol{\xi} - c_i \|\boldsymbol{\xi}\| - \mathbf{u}_i \cdot \boldsymbol{\xi})(\mathbf{u}_I \cdot \boldsymbol{\xi} - \mathbf{u}_i \cdot \boldsymbol{\xi}),$$

the expression for the eigenvector $\mathbf{W}_{\{\mathbf{u}_I \cdot \boldsymbol{\xi}\}}$ associated with $\mathbf{u}_I \cdot \boldsymbol{\xi}$ is

$$\begin{pmatrix} -K_1 K_2 \\ K_2 \left((\mathbf{u}_1 \cdot \boldsymbol{\xi} - \mathbf{u}_I \cdot \boldsymbol{\xi}) \rho_1 \left((\beta_1 P_1 + \kappa_1 \rho_1) \|\boldsymbol{\xi}\|^2 - \beta_1 \boldsymbol{\xi}^T \mathbf{P}_I \boldsymbol{\xi} \right) + \|\boldsymbol{\xi}\|^2 \left((P\mathbf{u})_I^T \boldsymbol{\xi} - \mathbf{u}_1^T \mathbf{P}_I \boldsymbol{\xi} \right) \right) \\ K_2 \left((P\mathbf{u})_I^T \boldsymbol{\xi} - \mathbf{u}_1^T \mathbf{P}_I \boldsymbol{\xi} - \rho_1 (\beta_1 P_1 + \rho_1 \kappa_1) (\mathbf{u}_I \cdot \boldsymbol{\xi} - \mathbf{u}_1 \cdot \boldsymbol{\xi}) \right) \left((\mathbf{u}_I \cdot \boldsymbol{\xi} - \mathbf{u}_1 \cdot \boldsymbol{\xi}) \boldsymbol{\xi} + \|\boldsymbol{\xi}\|^2 \mathbf{u}_1 \right) \\ + \rho_1 \beta_1 \left(c_1^2 \left(\boldsymbol{\xi} \boldsymbol{\xi}^T - \boldsymbol{\xi}^T \boldsymbol{\xi} \right) + (\mathbf{u}_I \cdot \boldsymbol{\xi} - \mathbf{u}_1 \cdot \boldsymbol{\xi}) \left(\mathbf{u}_I \cdot \boldsymbol{\xi} - \mathbf{u}_1 \cdot \boldsymbol{\xi} + \mathbf{u}_1 \boldsymbol{\xi}^T \right) \right) \mathbf{P}_I \boldsymbol{\xi} \\ K_2 \Big(\rho_1 \beta_1 \Big((\mathbf{u}_I \cdot \boldsymbol{\xi} - \mathbf{u}_1 \cdot \boldsymbol{\xi} - c_1 \|\boldsymbol{\xi}\|) (\mathbf{u}_I \cdot \boldsymbol{\xi} - \mathbf{u}_1 \cdot \boldsymbol{\xi} + c_1 \|\boldsymbol{\xi}\|) (P\mathbf{u})_I^T \boldsymbol{\xi} \\ + \left(c_1^2 \mathbf{u}_1 \cdot \boldsymbol{\xi} + H_1 (\mathbf{u}_I \cdot \boldsymbol{\xi} - \mathbf{u}_1 \cdot \boldsymbol{\xi}) \right) \boldsymbol{\xi}^T \mathbf{P}_I \boldsymbol{\xi} \Big) \\ - \Big(\rho_1 (\beta_1 P_1 + \rho_1 \kappa_1) (\mathbf{u}_I \cdot \boldsymbol{\xi} - \mathbf{u}_1 \cdot \boldsymbol{\xi}) + \mathbf{u}_1^T \mathbf{P}_I \boldsymbol{\xi} - (P\mathbf{u})_I^T \boldsymbol{\xi} \Big) \\ \Big(\mathbf{u}_1 \cdot \boldsymbol{\xi} (\mathbf{u}_I \cdot \boldsymbol{\xi} - \mathbf{u}_1 \cdot \boldsymbol{\xi}) + H_1 \|\boldsymbol{\xi}\|^2 \Big) \Big) \\ \dots \end{pmatrix},$$

where the dots represent the same expression by inverting the sign and the indices 1 and 2. As the first component of the other eigenvectors is always 0 (see the proof of Proposition 6.1), this vector is always independent of the other eigenvectors provided $K_i \neq 0$, which means that $\mathbf{u}_I \cdot \boldsymbol{\xi} \neq \mathbf{u}_i \cdot \boldsymbol{\xi}$ and $\mathbf{u}_I \cdot \boldsymbol{\xi} \neq \mathbf{u}_i \cdot \boldsymbol{\xi} \pm c_i \|\boldsymbol{\xi}\|$. This proves the first point of the proposition.
Suppose now that $\mathbf{u}_I \cdot \boldsymbol{\xi} = \mathbf{u}_1 \cdot \boldsymbol{\xi}$. Then in dimension d, this eigenvalue is of multiplicity $d+1$, with at least d eigenvectors, which are the usual eigenvectors associated with $\mathbf{u}_1 \cdot \boldsymbol{\xi}$.

$$\mathbf{W}^{\boldsymbol{\xi}}_{\{\mathbf{u}_1 \cdot \boldsymbol{\xi}\}} = \begin{pmatrix} 0 \\ 1 \\ \mathbf{u}_1 \\ H_1 - \rho_1 \beta_1 c_1^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \mathbf{W}^{\boldsymbol{\zeta}_j}_{\{\mathbf{u}_1 \cdot \boldsymbol{\xi}\}} = \begin{pmatrix} 0 \\ 0 \\ \boldsymbol{\zeta}_j \\ \mathbf{u}_1 \cdot \boldsymbol{\zeta}_j \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where the $\boldsymbol{\zeta}_j$ are $(d-1)$ orthonormal vectors which span the orthogonal of $\boldsymbol{\xi}$. We consider the following vector, obtained as the second term of the Taylor expansion of $\mathbf{W}_{\{\mathbf{u}_I \cdot \boldsymbol{\xi}\}}$ when $\mathbf{u}_I \cdot \boldsymbol{\xi} \to \mathbf{u}_1 \cdot \boldsymbol{\xi}$

$$\check{\mathbf{W}}_{\{\mathbf{u}_I \cdot \boldsymbol{\xi}\}} = \begin{pmatrix} -\check{K}_1 K_2 \\ K_2 \left(-\rho_1 ((\beta_1 P_1 + \kappa_1 \rho_1) \|\boldsymbol{\xi}\|^2 - \beta_1 \boldsymbol{\xi}^T \mathbf{P}_I \boldsymbol{\xi}) \right) \\ K_2 \left(((P\mathbf{u})_I \cdot \boldsymbol{\xi} \mathbf{u}_1^T \mathbf{P}_I \boldsymbol{\xi}) \boldsymbol{\xi} - \rho_1 (\beta_1 P_1 + \rho_1 \kappa_1) \|\boldsymbol{\xi}\|^2 \mathbf{u}_1 + \rho_1 \beta_1 \mathbf{u}_1 \boldsymbol{\xi}^T \mathbf{P}_I \boldsymbol{\xi} \right) \\ K_2 \left(\rho_1 \beta_1 H_1 \boldsymbol{\xi}^T \mathbf{P}_I \boldsymbol{\xi} - \rho_1 (\beta_1 P_1 + \rho_1 \kappa_1) H_1 \|\boldsymbol{\xi}\|^2 - \left(\mathbf{u}_1^T \mathbf{P}_I \boldsymbol{\xi} - (P\mathbf{u})_I \cdot \boldsymbol{\xi} \right) \mathbf{u}_1 \cdot \boldsymbol{\xi} \right) \\ \dots \end{pmatrix}$$

with $\check{K}_1 = \rho_1\beta_1(\mathbf{u}_I\cdot\boldsymbol{\xi} - \mathbf{u}_1\cdot\boldsymbol{\xi} + c_1\|\boldsymbol{\xi}\|)(\mathbf{u}_I\cdot\boldsymbol{\xi} - c_1\|\boldsymbol{\xi}\| - \mathbf{u}_1\cdot\boldsymbol{\xi})$, and where the dots are exactly the same terms as $\mathbf{W}_{\{\mathbf{u}_I\cdot\boldsymbol{\xi}\}}$, where K_1 was replaced by $\check{K}_1$. Then

$$\begin{aligned}\mathbf{J}_{\boldsymbol{\xi}}\check{\mathbf{W}}_{\{\mathbf{u}_I\cdot\boldsymbol{\xi}\}} = (\mathbf{u}_I\cdot\boldsymbol{\xi})\,\check{\mathbf{W}}_{\{\mathbf{u}_I\cdot\boldsymbol{\xi}\}} + \|\boldsymbol{\xi}\|^2\left(\mathbf{u}_1^T\mathbf{P}_I\boldsymbol{\xi} - (P\mathbf{u})_I\cdot\boldsymbol{\xi}\right)\mathbf{W}^{\boldsymbol{\xi}}_{\{\mathbf{u}_1\cdot\boldsymbol{\xi}\}}\\ +\sum_{j=1}^{d-1}\rho_1\beta_1c_1^2\left(\boldsymbol{\zeta}_j^T\mathbf{P}_I\boldsymbol{\xi}\right)\mathbf{W}^{\boldsymbol{\zeta}_j}_{\{\mathbf{u}_1\cdot\boldsymbol{\xi}\}}.\end{aligned}$$

This last equation gives in general a Jordan block in the matrix, except when $\boldsymbol{\zeta}_j^T\mathbf{P}_I\boldsymbol{\xi} = 0$ for all j and $\mathbf{u}_1^T\mathbf{P}_I\boldsymbol{\xi} - (P\mathbf{u})_I\cdot\boldsymbol{\xi} = 0$. These two conditions are therefore necessary and sufficient for ensuring hyperbolicity.

Suppose now that $\mathbf{u}_I\cdot\boldsymbol{\xi} = \mathbf{u}_1\cdot\boldsymbol{\xi} + c_1\|\boldsymbol{\xi}\|$. The eigenvector associated with $\mathbf{u}_1\cdot\boldsymbol{\xi} + c_1\|\boldsymbol{\xi}\|$ is

$$\mathbf{W}_{\{\mathbf{u}_1\cdot\boldsymbol{\xi}+c_1\}} = \begin{pmatrix} 0\\ 1\\ \mathbf{u}_1 + c_1\dfrac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\\ H_1 + c_1\dfrac{\mathbf{u}_1\cdot\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\\ 0\\ \vdots\\ 0\end{pmatrix}.$$

We consider the following vector, obtained as the second term of the Taylor expansion of $\mathbf{W}_{\{\mathbf{u}_I\cdot\boldsymbol{\xi}\}}$ when $\mathbf{u}_I\cdot\boldsymbol{\xi}\to\mathbf{u}_1\cdot\boldsymbol{\xi} + c_1\|\boldsymbol{\xi}\|$:

$$\check{\mathbf{W}}_{\{\mathbf{u}_I\cdot\boldsymbol{\xi}\}} = \begin{pmatrix} -\check{K}_1K_2\\ K_2\left(-\rho_1\left((\beta_1P_1+\kappa_1\rho_1)\|\boldsymbol{\xi}\|^2 - \beta_1\boldsymbol{\xi}^T\mathbf{P}_I\boldsymbol{\xi}\right)\right)\\ K_2\Big(\left((P\mathbf{u})_I\cdot\boldsymbol{\xi} - \mathbf{u}_1^T\mathbf{P}_I\boldsymbol{\xi}\right)\boldsymbol{\xi} - \rho_1(\beta_1P_1+\rho_1\kappa_1)\|\boldsymbol{\xi}\|^2\mathbf{u}_1\\ -\rho_1(\beta_1P_1+\rho_1\kappa_1)\left(\mathbf{u}_I\cdot\boldsymbol{\xi} - \mathbf{u}_1\cdot\mathbf{n} + c_1\|\boldsymbol{\xi}\|\right)\boldsymbol{\xi}\\ +\rho_1\beta_1\boldsymbol{\xi}^T\mathbf{P}_I\boldsymbol{\xi}\mathbf{u}_1 + \rho_1\beta_1\left(\mathbf{u}_I\cdot\boldsymbol{\xi} - \mathbf{u}_1\cdot\mathbf{n} + c_1\|\boldsymbol{\xi}\|\right)\mathbf{P}_I\boldsymbol{\xi}\Big)\\ K_2\Big(\rho_1\beta_1\left(\mathbf{u}_I\cdot\boldsymbol{\xi} - \mathbf{u}_1\cdot\boldsymbol{\xi} + c_1\|\boldsymbol{\xi}\|\right)(P\mathbf{u})_I\cdot\boldsymbol{\xi} + \rho_1\beta_1H_1\boldsymbol{\xi}^T\mathbf{P}_I\boldsymbol{\xi}\\ -\left(\mathbf{u}_1^T\mathbf{P}_I\boldsymbol{\xi} - (P\mathbf{u})_I\cdot\boldsymbol{\xi}\right)\mathbf{u}_1\cdot\boldsymbol{\xi}\\ -\rho_1\left(\beta_1P_1+\rho_1\kappa_1\right)\Big(\mathbf{u}_1\cdot\boldsymbol{\xi}\left(\mathbf{u}_I\cdot\boldsymbol{\xi} - \mathbf{u}_1\cdot\boldsymbol{\xi} + c_1\|\boldsymbol{\xi}\|\right) + H_1\|\boldsymbol{\xi}\|^2\Big)\Big)\\ \cdots\end{pmatrix}$$

with $\check{K}_1 = \rho_1\beta_1(\mathbf{u}_I\cdot\boldsymbol{\xi} - \mathbf{u}_1\cdot\boldsymbol{\xi} + c_1\|\boldsymbol{\xi}\|)(\mathbf{u}_I\cdot\boldsymbol{\xi} - \mathbf{u}_1\cdot\boldsymbol{\xi})$, and where the dots are exactly the same terms as $\mathbf{W}_{\{\mathbf{u}_I\cdot\boldsymbol{\xi}\}}$, where K_1 was replaced by $\check{K}_1$. Then

$$\begin{aligned}&\mathbf{J}_{\boldsymbol{\xi}}\check{\mathbf{W}}_{\{\mathbf{u}_I\cdot\boldsymbol{\xi}\}} = (\mathbf{u}_I\cdot\boldsymbol{\xi})\,\check{\mathbf{W}}_{\{\mathbf{u}_I\cdot\boldsymbol{\xi}\}}\\ &\quad +K_2\left(-\left(\rho_1c_1\left(P_1\beta_1+\kappa_1\rho_1\right)\|\boldsymbol{\xi}\|^2 - \beta_1\boldsymbol{\xi}^T\mathbf{P}_I\boldsymbol{\xi}\right) + \|\boldsymbol{\xi}\|\left((P\mathbf{u})_I\cdot\boldsymbol{\xi} - \mathbf{u}_1^T\mathbf{P}_I\boldsymbol{\xi}\right)\right)\|\boldsymbol{\xi}\|\,\mathbf{W}_{\{\mathbf{u}_1\cdot\boldsymbol{\xi}+c_1\}}.\end{aligned}$$

This means that except when

$$-\left(\rho_1c_1\left(P_1\beta_1+\kappa_1\rho_1\right)\|\boldsymbol{\xi}\|^2 - \beta_1\boldsymbol{\xi}^T\mathbf{P}_I\boldsymbol{\xi}\right) + \|\boldsymbol{\xi}\|\left((P\mathbf{u})_I\cdot\boldsymbol{\xi} - \mathbf{u}_1^T\mathbf{P}_I\boldsymbol{\xi}\right) = 0,$$

the system is resonant. This ends the proof.

Appendix C. Details of the different proofs for entropy dissipation.

C.1. Proof of Proposition 6.4. All the proofs for the entropy dissipation of the nonconservative contribution rely on the following lemma.

LEMMA 1. *We suppose that the equation of state $(\tau, s) \mapsto p(\tau, s)$ of the fluid k has the following properties:*

$$\frac{\partial p}{\partial \tau} \leq \frac{c}{\tau^\alpha}, \alpha > 2 \quad \frac{\partial p}{\partial s} > 0 \quad \frac{\partial^2 p}{\partial \tau^2} > 0. \tag{C.1}$$

We denote by $P^\star_{\bar{k}k}$ and $u^\star_{\bar{k}k}$ the pressure and velocity of the solution of the Riemann problem with a right state having P_k and $\mathbf{u}_k \cdot \mathbf{n}$ as pressure and velocity. Then for any unitary direction $\mathbf{n}$

$$\left(P^\star_{\bar{k}k} - P_k\right)\left(u^\star_{\bar{k}k} - \mathbf{u}_k \cdot \mathbf{n}\right) \geq 0.$$

Proof. Following [22, Theorem 3.1 p. 134], conditions (C.1) ensure that the wave curve coming from the point $(\mathbf{u}_k \cdot \mathbf{n}, P_k)$ is increasing. This means that an increasing function Φ exists such that all the states $(u^\star, P^\star)$ that can be linked with the state $(\mathbf{u}_k \cdot \mathbf{n}, P_k)$ through a right wave curve follow

$$\mathbf{u}^\star = \mathbf{u}_k \cdot \mathbf{n} + \Phi(P^\star).$$

The function Φ being such that $\Phi(P_k) = 0$, we find

$$\left(P^\star_{\bar{k}k} - P_k\right)\left(u^\star_{\bar{k}k} - \mathbf{u}_k \cdot \mathbf{n}\right) = \left(P^\star_{\bar{k}k} - P_k\right)\left(\Phi(P^\star_{\bar{k}k}) - \Phi(P_k)\right),$$

which is nonnegative, because Φ is increasing.
□

Note that the conditions (C.1) can probably be relaxed; see [35] for tighter conditions on the equation of state.

We now prove Proposition 6.4 for the long memory model described in Proposition 5.2. In this case, we have

$$\begin{aligned}
&\frac{1}{T_k}\left((P\mathbf{u})_I + P_k\mathbf{u}_k - P_k\mathbf{u}_I - \mathbf{u}_k\mathbf{P}_I\right)\nabla\alpha_k \\
&= \frac{1}{T_k}\left(P^\star_{\bar{k}k}\left(\frac{\nabla\alpha_k(\mathbf{x})}{\|\nabla\alpha_k(\mathbf{x})\|}\right) - P_k\right)\left(u^\star_{\bar{k}k}\left(\frac{\nabla\alpha_k(\mathbf{x})}{\|\nabla\alpha_k(\mathbf{x})\|}\right) - \mathbf{u}_k \cdot \frac{\nabla\alpha_k(\mathbf{x})}{\|\nabla\alpha_k(\mathbf{x})\|}\right)\|\nabla\alpha_k(\mathbf{x})\|.
\end{aligned}$$

Applying Lemma 1 gives immediately the entropy dissipation.

We now focus on the second part of the proposition. If the nonconservative part of the entropy contribution is 0, then

$$(P\mathbf{u})_I + P_k\mathbf{u}_k - P_k\mathbf{u}_I - \mathbf{u}_k P_I = 0 \tag{C.2}$$

for $k = 1, 2$. If the velocity is such that $\mathbf{u}_I = Y_k\mathbf{u}_k + Y_{\bar{k}}\mathbf{u}_{\bar{k}}$, then equaling the two formulas for $(P\mathbf{u})_I$ obtained from (C.2) leads to $P_I = Y_{\bar{k}}P_k + Y_k P_{\bar{k}}$. Using this last expression for P_I leads to $(P\mathbf{u})_I = Y_k P_{\bar{k}}\mathbf{u}_k + Y_{\bar{k}}P_k\mathbf{u}_{\bar{k}}$.

C.2. Proof of Proposition 6.5. We first prove the following lemma

LEMMA 2. *With the same the hypothesis and notations as Lemma 1, for any unitary vector $\mathbf{n}$, defining*

$$\Psi(\mathbf{n}) := P_k u^\star_{\bar{k}k}(\mathbf{n}) + P^\star_{\bar{k}k}(\mathbf{n})(\mathbf{u}_k \cdot \mathbf{n}) - P^\star_{\bar{k}k}(\mathbf{n})u^\star_{\bar{k}k}(\mathbf{n}),$$

and defining $\tilde{\Psi}(\mathbf{n}) := \Psi(\mathbf{n}) + \Psi(-\mathbf{n})$, *then*

$$\tilde{\Psi}(\mathbf{n}) \leq 0.$$

Proof.

$$\begin{aligned}
\tilde{\Psi}(\mathbf{n}) &= P_k u^\star_{\bar{k}k}(\mathbf{n}) + P^\star_{\bar{k}k}(\mathbf{n})(\mathbf{u}_k \cdot \mathbf{n}) - P^\star_{\bar{k}k}(\mathbf{n}) u^\star_{\bar{k}k}(\mathbf{n}) \\
&\quad + P_k u^\star_{\bar{k}k}(-\mathbf{n}) + P^\star_{\bar{k}k}(-\mathbf{n})\mathbf{u}_k \cdot (-\mathbf{n}) - P^\star_{\bar{k}k}(-\mathbf{n}) u^\star_{\bar{k}k}(-\mathbf{n}) \\
&= P_k u^\star_{\bar{k}k}(\mathbf{n}) + P^\star_{\bar{k}k}(\mathbf{n})(\mathbf{u}_k \cdot \mathbf{n}) - P^\star_{\bar{k}k}(\mathbf{n}) u^\star_{\bar{k}k}(\mathbf{n}) \\
&\quad - P_k u^\star_{k\bar{k}}(\mathbf{n}) - P^\star_{k\bar{k}}(\mathbf{n})(\mathbf{u}_k \cdot \mathbf{n}) + P^\star_{k\bar{k}}(\mathbf{n}) u^\star_{k\bar{k}}(\mathbf{n}) \\
\tilde{\Psi}(\mathbf{n}) &= (P_k - P^\star_{\bar{k}k}) u^\star_{\bar{k}k}(\mathbf{n}) + (\mathbf{u}_k \cdot \mathbf{n}) \left(P^\star_{\bar{k}k}(\mathbf{n}) - P^\star_{k\bar{k}}(\mathbf{n}) \right) \\
&\quad + u^\star_{k\bar{k}}(\mathbf{n}) \left(P^\star_{k\bar{k}}(\mathbf{n}) - P_k \right).
\end{aligned}$$

Then, as said in the proof of Lemma 1, an increasing function Φ exists such that

$$\begin{cases} u^\star_{\bar{k}k}(\mathbf{n}) = \mathbf{u}_k \cdot \mathbf{n} + \Phi(P^\star_{\bar{k}k}), \\ u^\star_{k\bar{k}}(\mathbf{n}) = \mathbf{u}_k \cdot \mathbf{n} - \Phi(P^\star_{k\bar{k}}), \end{cases}$$

with $\Phi(P_k) = 0$. This leads to

$$\begin{aligned}
\tilde{\Psi}(\mathbf{n}) &= (P_k - P^\star_{\bar{k}k}) u^\star_{\bar{k}k}(\mathbf{n}) + (\mathbf{u}_k \cdot \mathbf{n}) \left(P^\star_{\bar{k}k}(\mathbf{n}) - P^\star_{k\bar{k}}(\mathbf{n}) \right) \\
&\quad + u^\star_{k\bar{k}}(\mathbf{n}) \left(P^\star_{k\bar{k}}(\mathbf{n}) - P_k \right) \\
&= (P_k - P^\star_{\bar{k}k}) \left(\mathbf{u}_k \cdot \mathbf{n} + \Phi(P^\star_{\bar{k}k}(\mathbf{n})) \right) \\
&\quad + (\mathbf{u}_k \cdot \mathbf{n}) \left(P^\star_{\bar{k}k}(\mathbf{n}) - P^\star_{k\bar{k}}(\mathbf{n}) \right) \\
&\quad + \left(\mathbf{u} \cdot \mathbf{n} - \Phi(P^\star_{k\bar{k}})(\mathbf{n}) \right) \left(P^\star_{k\bar{k}}(\mathbf{n}) - P_k \right) \\
&= (P_k - P^\star_{\bar{k}k}) \Phi \left(P^\star_{\bar{k}k}(\mathbf{n}) \right) - \Phi \left(P^\star_{k\bar{k}}(\mathbf{n}) \right) \left(P^\star_{k\bar{k}}(\mathbf{n}) - P_k \right) \\
&= (P_k - P^\star_{\bar{k}k}) \left(\Phi \left(P^\star_{\bar{k}k}(\mathbf{n}) \right) - \Phi \left(P_k \right) \right) \\
&\quad + \left(\Phi(P_k) - \Phi \left(P^\star_{k\bar{k}}(\mathbf{n}) \right) \right) \left(P^\star_{k\bar{k}}(\mathbf{n}) - P_k \right).
\end{aligned}$$

As Φ is increasing, this leads to $\tilde{\Psi}(\mathbf{n}) \leq 0$, which ends the proof. ∎

We now prove Proposition 6.5 for the nonlinear relaxation terms of Proposition 5.3

$$\begin{aligned}
&-P_k \mathcal{R}^{(\alpha)}_k + \mathbf{u}_k \cdot \mathcal{R}^{(\rho\mathbf{u})}_k - \mathcal{R}^{(\rho E)}_k \\
&= \lambda_1 \mathrm{e}^{-\frac{\partial_x m_k(x)^2}{2\lambda_1^2}} \frac{\mathrm{e}^{-\frac{m_k(x)^2}{2}}}{2\pi} \tilde{\Psi}(\partial_x \alpha_k(x)).
\end{aligned}$$

where $\tilde{\Psi}$ is the function defined in Lemma 2. Applying this lemma ends the proof.

We are now interested in proving Proposition 6.5 with the nonlinear relaxation terms of Proposition 5.4. In this case

$$\begin{aligned}
&-P_k \mathcal{R}^{(\alpha)}_k + \mathbf{u}_k \cdot \mathcal{R}^{(\rho\mathbf{u})}_k - \mathcal{R}^{(\rho E)}_k \\
&= \frac{\exp(-m_k(\mathbf{x})^2/2)}{2\pi \left| \mathbb{V}_{d-1} \right| \left| \det \check{\Sigma} \right|^{1/2}} \int_{\mathbf{x}_d \in Q^T \Lambda(\mathbb{S}^{d-1})} \Psi \left(\frac{\mathbf{x}_d}{\|\mathbf{x}_d\|} \right) \|\mathbf{x}_d\| \, \mathrm{d}\mathbf{x}_d,
\end{aligned}$$

where Ψ was defined in Lemma 2. The integral can be divided as the sum of two integrals

$$\begin{aligned}
\int_{\mathbf{x}_d \in Q^T \Lambda(\mathbb{S}^{d-1})} \Psi \left(\frac{\mathbf{x}_d}{\|\mathbf{x}_d\|} \right) \|\mathbf{x}_d\| \, \mathrm{d}\mathbf{x}_d &= \int_{\mathbf{x}_d \in Q^T \Lambda(\mathbb{S}^{d-1}), \mathbf{x}^{(0)}_d \geq 0} \Psi \left(\frac{\mathbf{x}_d}{\|\mathbf{x}_d\|} \right) \|\mathbf{x}_d\| \, \mathrm{d}\mathbf{x}_d \\
&\quad + \int_{\mathbf{x}_d \in Q^T \Lambda(\mathbb{S}^{d-1}), \mathbf{x}^{(0)}_d \leq 0} \Psi \left(\frac{\mathbf{x}_d}{\|\mathbf{x}_d\|} \right) \|\mathbf{x}_d\| \, \mathrm{d}\mathbf{x}_d.
\end{aligned}$$

Remarking that the second integral includes exactly the opposite vectors of the first one, we find

$$\int_{\mathbf{x}_d \in Q^T \Lambda(\mathbb{S}^{d-1})} \Psi\left(\frac{\mathbf{x}_d}{\|\mathbf{x}_d\|}\right) \|\mathbf{x}_d\| \, \mathrm{d}\mathbf{x}_d = \int_{\mathbf{x}_d \in Q^T \Lambda(\mathbb{S}^{d-1}), \mathbf{x}_d^{(0)} \geq 0} \tilde{\Psi}\left(\frac{\mathbf{x}_d}{\|\mathbf{x}_d\|}\right) \|\mathbf{x}_d\| \, \mathrm{d}\mathbf{x}_d.$$

Applying Lemma 2 ends the proof.

Last, we prove Proposition 6.5 for the models proposed in Proposition 5.5 and Proposition 5.6. In this case

$$\begin{aligned}
&\frac{1}{T_k}\left(-P_k \mathcal{R}_k^{(\alpha)} + \mathbf{u}_k \cdot \mathcal{R}_k^{(\rho \mathbf{u})} - \mathcal{R}_k^{(\rho E)}\right) \\
&= \frac{1}{T_k}\left(\left(P_I^L - P_k\right) \mathcal{R}_k^{(\alpha)} + \left(\mathbf{u}_k - \mathbf{u}_I^L\right) \cdot \mathcal{R}_k^{(\rho \mathbf{u})}\right) \\
&= -\frac{Z_k \exp(-m_k(\mathbf{x})^2/2) \left|\mathbb{S}^{d-1}\right|}{2\pi \left|\mathbb{V}_{d-1}\right| (Z_k + Z_{\bar{k}})^2} \left(\mathscr{L}(\Lambda)(P_{\bar{k}} - P_k)^2 + Z_{\bar{k}}^2 (\mathbf{u}_{\bar{k}} - \mathbf{u}_k) \tilde{\Lambda} (\mathbf{u}_{\bar{k}} - \mathbf{u}_k)\right).
\end{aligned}$$

As $\mathscr{L}(\Lambda)$ is nonnegative and $\tilde{\Lambda}$ is also nonnegative, we get

$$\frac{1}{T_k}\left(-P_k \mathcal{R}_k^{(\alpha)} + \mathbf{u}_k \cdot \mathcal{R}_k^{(\rho \mathbf{u})} - \mathcal{R}_k^{(\rho E)}\right) \leq 0,$$

and this ends the proof of Proposition 6.5.

Appendix D. Proof of Proposition 6.6. The computations for this section are not straightforward, and *SageMath* [46] was used for solving formally some linear systems. In general, a symmetrizer for (6.1) can be written blockwise

$$\mathcal{S} = \begin{pmatrix} s_{11} & \bar{s}_{12}^T & \bar{s}_{13}^T \\ \bar{s}_{12} & S_{22} & S_{23} \\ \bar{s}_{13} & S_{23}^T & S_{33} \end{pmatrix},$$

where s_{11} is a scalar, $\bar{s}_{12}$ and $\bar{s}_{13}$ are vectors of size $(d+2)$, and S_{22}, S_{23} and S_{33} are $(d+2) \times (d+2)$ matrices, and S_{22} and S_{33} are symmetric. Writing blockwise that $\mathcal{S}$ is a symmetrizer of (6.1) immediately gives that S_{ii} is a symmetrizer of $\mathrm{Euler}_{i-1}(\boldsymbol{\xi})$ for $i = 2$ and 3. In general, symmetrizers for the Euler system are a two parameters family of matrices, which reads in conservative variables

$$\begin{pmatrix} s_E^{11} & -\left(\rho\beta s_E^{(2)} + s_E^{(1)}\right) \mathbf{u}^T & s_E^{(1)} \\ -\left(\rho\beta s_E^{(2)} + s_E^{(1)}\right) \mathbf{u} & \rho\beta s_E^{(2)} \mathrm{I}_d + s_E^{(0)} \mathbf{u}^T \mathbf{u} & -s_E^{(1)} \mathbf{u} \\ s_E^{(1)} & -s_E^{(1)} \mathbf{u}^T & s_E^{(0)} \end{pmatrix}$$

with

$$s_E^{11} = s_E^{(2)}\left(\rho\beta\left(|\mathbf{u}|^2 + c^2\right) + |\mathbf{u}|^2 - H\right) + s_E^{(1)}\left(|\mathbf{u}|^2 - H\right),$$

and where $s_E^{(0)}$ and $s_E^{(1)}$ are two free parameters, and $s_E^{(2)} = s_E^{(1)} - \left(|\mathbf{u}|^2 - H\right) s_E^{(0)}$. Some tedious computation leads also to $S_{23} = 0$. If (6.1) is formally rewritten blockwise as

$$\begin{pmatrix} \mathbf{u}_I \cdot \boldsymbol{\xi} & 0 & 0 \\ \mathrm{j}_1(\boldsymbol{\xi}) & \mathrm{Euler}_1(\boldsymbol{\xi}) & 0 \\ \mathrm{j}_2(\boldsymbol{\xi}) & 0 & \mathrm{Euler}_2(\boldsymbol{\xi}) \end{pmatrix},$$

then $\mathcal{S}$ symmetrizes $J_{\boldsymbol{\xi}}$ if and only if

$$\begin{cases} \left(\bar{s}_{12}^T \mathrm{Euler}_1(\boldsymbol{\xi})\right)^T = (\mathbf{u}_I \cdot \boldsymbol{\xi})\, \bar{s}_{12} + S_{22}\mathrm{j}_1(\boldsymbol{\xi}), \\ \left(\bar{s}_{13}^T \mathrm{Euler}_2(\boldsymbol{\xi})\right)^T = (\mathbf{u}_I \cdot \boldsymbol{\xi})\, \bar{s}_{13} + S_{33}\mathrm{j}_2(\boldsymbol{\xi}), \end{cases}$$

which are two independent systems involving either phase 1 or phase 2. We deal with the phase 1 and remove any index identifying the phases. Then j is solution of the system

$$A\mathrm{j} = b, \tag{D.1}$$

where the matrix A is

$$\begin{pmatrix} -(\mathbf{u}_I \cdot \boldsymbol{\xi}) & \left(c^2 + \dfrac{|\mathbf{u}|^2 - H}{\rho\beta}\right)\boldsymbol{\xi}^T - (\mathbf{u}\cdot\boldsymbol{\xi})\mathbf{u}^T & \left(c^2 - H + \dfrac{|\mathbf{u}|^2 - H}{\rho\beta}\right)(\mathbf{u}\cdot\boldsymbol{\xi}) \\ \boldsymbol{\xi} & ((\mathbf{u}\cdot\boldsymbol{\xi}) - (\mathbf{u}_I\cdot\boldsymbol{\xi}))\,\mathrm{I}_d + \boldsymbol{\xi}\mathbf{u}^T - \dfrac{\mathbf{u}\boldsymbol{\xi}^T}{\rho\beta} & H\boldsymbol{\xi} - \dfrac{(\mathbf{u}\cdot\boldsymbol{\xi})}{\rho\beta}\,\mathbf{u} \\ 0 & \dfrac{\boldsymbol{\xi}^T}{\rho\beta} & \dfrac{\mathbf{u}\cdot\boldsymbol{\xi}}{\rho\beta} + (\mathbf{u}\cdot\boldsymbol{\xi}) - (\mathbf{u}_I\cdot\boldsymbol{\xi}) \end{pmatrix},$$

and the right hand side is

$$b = \begin{pmatrix} \left((1+\rho\beta)\,\boldsymbol{\xi}^T\mathbf{P}_I\mathbf{u} - (P\mathbf{u})_I\cdot\boldsymbol{\xi} - \rho\,(P\beta + \kappa\rho)\,(\mathbf{u}\cdot\boldsymbol{\xi})\right) s_E^{(2)} \\ + \left(\boldsymbol{\xi}^T\mathbf{P}_I\mathbf{u} - (P\mathbf{u})_I\cdot\boldsymbol{\xi}\right)\left(|\mathbf{u}|^2 - H\right) s_E^{(0)} \\ \rho\left((P\beta + \rho\kappa)\,\boldsymbol{\xi} - \beta\mathbf{P}_I\boldsymbol{\xi}\right) s_E^{(2)} - \left(\boldsymbol{\xi}^T\mathbf{P}_I\mathbf{u} - (P\mathbf{u})_I\cdot\boldsymbol{\xi}\right) s_E^{(0)}\mathbf{u} \\ \left(\boldsymbol{\xi}^T\mathbf{P}_I\mathbf{u} - (P\mathbf{u})_I\cdot\boldsymbol{\xi}\right) s_E^{(0)} \end{pmatrix}.$$

The system (D.1) is solved with Cramer's rule, which involves the determinant of A

$$\det A = (\mathbf{u}\cdot\boldsymbol{\xi} - \mathbf{u}_I\cdot\boldsymbol{\xi})^d\,(\mathbf{u}_I\cdot\boldsymbol{\xi} - \mathbf{u}\cdot\boldsymbol{\xi} - c\,\|\boldsymbol{\xi}\|)\,(\mathbf{u}_I\cdot\boldsymbol{\xi} - \mathbf{u}\cdot\boldsymbol{\xi} + c\,\|\boldsymbol{\xi}\|)\,.$$

We are interested in the last component of j. This component can be expressed with Cramer's rule, as the ratio of a determinant by the determinant of A. In general, it can be written as

$$\mathrm{j}_{d+2} = \frac{Q(\mathbf{u}_I\cdot\boldsymbol{\xi})(\mathbf{u}_I\cdot\boldsymbol{\xi} - \mathbf{u}\cdot\boldsymbol{\xi})^{d-1}}{\det A},$$

where Q is a polynomial. As the symmetrizer should be independent of $\boldsymbol{\xi}$, Q should have $\mathbf{u}\cdot\boldsymbol{\xi}$ and $\mathbf{u}\cdot\boldsymbol{\xi} \pm c\,\|\boldsymbol{\xi}\|$ as single roots. The full expression of Q is not given here, but only its values for $\mathbf{u}_I\cdot\boldsymbol{\xi} = \mathbf{u}\cdot\boldsymbol{\xi} \pm c\,\|\boldsymbol{\xi}\|$:

$$\begin{aligned} Q(\mathbf{u}\cdot\boldsymbol{\xi} \pm c\,\|\boldsymbol{\xi}\|) = &\left(\|\boldsymbol{\xi}\|\left(\boldsymbol{\xi}^T\mathbf{P}_I\mathbf{u} - (P\mathbf{u})_I\cdot\boldsymbol{\xi}\right) \pm \rho c\left((P\beta + \rho\kappa)\,\|\boldsymbol{\xi}\|^2 - \beta\boldsymbol{\xi}^T\mathbf{P}_I\boldsymbol{\xi}\right)\right) \\ &\times\left(s_E^{(1)} - \left(|\mathbf{u}|^2 - H\right) s_E^{(0)}\right)\|\boldsymbol{\xi}\|\,. \end{aligned} \tag{D.2}$$

Equations (D.2) induces immediately

$$s_E^{(1)} = \left(|\mathbf{u}|^2 - H\right) s_E^{(0)}, \tag{D.3}$$

which can also be seen as $s_E^{(2)} = 0$. Now, remarking that under condition (D.3), b is an eigenvector associated with the eigenvalue $\mathbf{u}\cdot\boldsymbol{\xi} - \mathbf{u}_I\cdot\boldsymbol{\xi}$, we find

$$\mathrm{j} = \frac{\boldsymbol{\xi}^T\mathbf{P}_I\mathbf{u} - (P\mathbf{u})_I\cdot\boldsymbol{\xi}}{\mathbf{u}\cdot\boldsymbol{\xi} - \mathbf{u}_I\cdot\boldsymbol{\xi}} \begin{pmatrix} |\mathbf{u}|^2 - H \\ -\mathbf{u} \\ 1 \end{pmatrix} s_E^{(0)}.$$

But as the symmetrizer should be independent of $\boldsymbol{\xi}$, a symmetrizer may exist only if

$$\exists M \quad \forall\boldsymbol{\xi} \qquad \boldsymbol{\xi}^T\mathbf{P}_I\mathbf{u} - (P\mathbf{u})_I\cdot\boldsymbol{\xi} = M\left(\mathbf{u}\cdot\boldsymbol{\xi} - \mathbf{u}_I\cdot\boldsymbol{\xi}\right).$$

However, we immediately see that even if this condition holds, the symmetrizer found cannot be full rank, and so cannot be strictly nonnegative, which means that the system is not symmetrizable.

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